MATH 6455
Differential Geometry
Spring Semester 2025

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Contents

Preface

These notes are for MATH 6455 Differential Geometry offered in the spring semester 2026 at Georgia Tech. The main objective is to cover some material from Heinz Hopf's classical lecture notes *Differential Geometry in the Large.* I would like especially to cover the foundational material for and the proof of Alexandrov's theorem on compact constant mean curvature surfaces embedded in $\mathbb{R}^3$. I will also try to cover Hopf's theorem on constant mean curvature immersions of the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ and some material on singularities of line fields and quadratic differentials.

Naturally I intended to start the course with a review of calculus and some material from an undergraduate course on the differential geoometry of curves and surfaces. I had five sources[1] on hand: Notes from the lectures of Heinz Hopf [3] mentioned above, Spivak's *A Comprehensive Introduction to Differential Geoemtry* [6], do Carmo's *Differential Geometry of Curves and Surfaces* [1], the book *Riemannian Geometry* by Gallot, Hulin and Lafontaine [2], and Barret O'Neill's *Elementary Differential Geometry* [5]. Each offered a different definition of surface.

It was clear to me Hopf's definitions, one for local differential geometry and one for the global differential geometry of compact surfaces, were ambiguous, somewhat informal, somewhat too restrictive, or perhaps all three. In particular, the proof of Hopf's first theorem of local differential geometry seemed to be incorrect as given—at least depending on the question of a topology for his "surface." At any rate, this posed something of a problem with using Hopf's notes as a primary reference.

O'Neill helpfully pointed out that the "main weakness of classical differential geometry was its lack of any adequate definition of *surface*." Gallot, Hulin, and Lafontaine state a proposition asserting the equivalence of three different definitions, so I thought it might be nice (and useful for the students

[1] among many others

in my course) to undertake giving a proof of that proposition and throw in the other four (or five including one of my own) definitions as well.

This accounts for a large portion of my chapter on “surfaces” below.

Part I

Alexandrov's Theorem

I would like to give a detailed proof of the following result:

Theorem 1. (A.D. Alexandrov) A compact, connected surface (without boundary) of constant mean curvture embedded in $\mathbb{R}^3$ is a (round) sphere.

Historical/bibliographical

A.D. Alexandrov (1912-1999) gave a lecture sketching his proof of the result in Zurich in 1955.

Alexandrov's technique has come to be known generally as the "method of moving planes."

Heinz Hopf (1894-1971) attended Alexandrov's lecture and gave a series of lectures in 1956 at Stanford including his exposition of Alexandrov's proof.

J.W. Gray (1932-2017) took notes on Hopf's lectures as a graduate student. These notes were apparently somewhat widely circulated though not published formally until 1983.

A little later Alexandrov published

(1956) *Uniqueness theorems for surfaces in the large, I.* Vestnik, Leningrad University **11**(19) pp. 5–17 (in Russian)
(1958) *Uniqueness theorems for surfaces in the large, V.* Vestnik, Leningrad University Ser. Mat. Meh. Astronom. **13** pp. 5–8 (in Russian)

I believe the first paper focused more on the comparison principle or the "PDE part" of the proof while the 1958 paper discussed the actual method of moving planes.

(1962) AMS Translation **21** (2) of *Uniqueness theorems for surfaces in the large, V.*

Alexandrov mentions in the 1962 translation paper (presumably the same as the 1958 paper) that he is (still) only outlining the proofs and plans to "present the complete proofs in later papers." At any rate the basic method of moving planes is described in this paper.

Henry Wente (1936-2020) published

(1980) *The symmetry of sessile and pendant drops,* Pacific J. **88** (2)

This is the exposition where I first read about Alexandrov's method.

(1983) *Differential Geometry in the Large,* Springer Lecture Notes

This volume containing the notes of John Gray was presumably published at the instigation of S.S. Chern (1911-2004) who wrote the preface. The notes of course were attributed to Heinz Hopf, though obviously he had been dead for more than a decade.

It is quite interesting that some details of Hopf's version of the method of moving planes as explained in *Differential Geometry in the Large* are much more complicated than the exposition/outline of Alexandrov or the exposition of Wente.

Around 1995 when I was working on *Symmetry via spherical reflection and spanning drops in a wedge,* Pacific J. Math. **180** (2), 1997, I went through the details of the reflection procedure carefully, and I think I filled in some of the details from the expositions of Alexandrov and Wente. I would like to go through the details of the more complicated version of Heinz Hopf, and see if the extra complication is really necessary. At any rate I hope my version of the details is correct, and there is clearly a lot of interesting differential geometry to be learnt by going through Hopf's exposition/Gray's notes.

Course Activities

While I will cover a reasonable amount of material on surface geometry, calculus, and maximum/comparison principles related to reflection, you may want to (and probably should) work on/think about some additional topics of your own interest in differential geometry. In particular, my work this semester will be somewhat light on the linear algebra and manifold theory side, but you should feel free to focus on topics in that other direction.

I would like for each student to make either one large "project" presentation during the semester or else a number of smaller in class presentations roughly equaling a one hour or hour and 10 minute presentation. There can be many options for this. Certainly presenting a number of "exercises" from my notes should be a possibility.

Alternatively, you can pick up a differential geometry book and make a presentation on an interesting section or topic from that book. Heinz Hopf's book is an obvious suggestion, as I will already be covering a large amount of material from there. The first part of Hopf's notes which is comprised of notes taken by Peter Lax is very interesting, and there is material on "fields of line elements" in both parts I've always found quite inspiring.

There are also probably some other topics I can suggest that might make good projects. I can think of one which really is a calculus project having to do with

- higher order reflection,
- extension of functions with regularity, and
- Whitney's extension theorem,

depending on how deep/far one wants to get into it. The last topic of the Whitney extension theorem also relates to some nice and worthwhile topics in the calculus of functions on Banach spaces—which relates both to geometry, calculus, and linear algebra/functional analysis. Basically, one could try to understand the proof of the Whitney extension theorem in the first volume of Hörmander's book on *The Analysis of Linear Partial Differential Operators.*

I will also post a list of papers I can recommend for you to read and upon which you can make a presentation. This is perhaps a good place for that list, so I'll start it now:

Robert Osserman, *Curvature in the eighties,* American Mathematical Monthly **97**(8) pp. 731–756 (1990)

Richard M. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces,* J. Differential Geom. **18**(4) pp. 791–809 (1983)

Meeks, Pérez, Ros, *Uniqueness of the Riemann minimal examples,* Invent. Math. **133**(1) pp. 107–132 (1998).

Hoffman, Karcher, *Complete embedded minimal surfaces of finite total curvature,* Eccyclopaedia Math. Sci., **90**, pp 5–93 (1997).

Hoffman, Karcher, Wei, *The singly periodic genus-one helicoid,* Comment. Math. Helv. **74**(2) pp. 248–279 (1999)

Blaine Lawson, *Complete minimal surfaces in* $\mathbb{S}^3$*,* Ann. of Math. **92**(2) pp. 335–374 (1970)

Robert Bryant, *Minimal surfaces of constant curvature in* $\mathbb{S}^n$*,* Trans. Amer. Math. Soc. **290**(1) pp. 259–271 (1985)

Simon Brendle, *Embedded minimal tori in* $\mathbb{S}^3$ *and the Lawson conjecture,* Acta Mathematica **211**(2) pp. 177–190 (2013)

Marques and Neves, *Min-max theory and the Willmore conjecture,* Annals of Mathematics **179**(2) pp. 683–782 (2014)

Vocabulary

In order to understand Alexandrov's theorem, or even the statement of Alexandrov's theorem, there are some words/concepts/notions one needs to understand. Here is a short list/outline of some of these words—and a couple extras.

1. surface
   - (a) compact
   - (b) connected
   - (c) embedded
2. mean curvature
   - (a) constant mean curvature
   - (b) minimal surface

Pieces of the proof

Heinz Hopf viewed Alexandrov's proof as having two parts, a geometric part and an analytic part. I think by the analytic part, he meant primarily the use of PDE and the maximum principle. For the geometric part I guess he meant both showing symmetry, by the moving planes method—which uses the maximum/comparison principle—and the characterization of the sphere in terms of the symmetry obtained by the moving planes method.

This doesn't seem like such a clean interpretation to me. I find it helpful to think of basically three parts: There is the moving planes procedure proper, which involves the geometric results of reflecting a surface across a moving plane. Then there is the application of the comparison principle to obtain certain symmetry properties for the surface. Then there is a third part which may be viewed as showing the symmetry of the surface which has been established forces the surface to be a round sphere.

A careful detailed proof of Alexandrov's theorem, the kind at which I aim, involves a good deal of calculus, differential geometry of surfaces, and partial differential equations. The required linear algebra and manifold theory is less substantial. A natural extended outline is the following:

1. surfaces and the reflection of surfaces

2. maximum and comparison principles (Eberhard Hopf and A.D. Alexandrov)

3. mean curvature and the relation to elliptic PDE

4. showing symmetry

5. symmetric surfaces and the consequences of symmetry

6. Hopf's exceptional directions.

Hopefully there will be time also for a discussion of Heinz Hopf's theorem on immersed surfaces of constant mean curvature and a couple other topics involving holomorphic structure, as well as some material from the calculus of variations which fits in naturally.

Chapter 1

Calculus

1.1 Sets

Given $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_j \in \mathbb{R},\ j = 1, 2, \ldots, n\}$$

Even sometimes $\mathbb{R}^0 = \{0\}$ is some kind of "canonical" set with a single point—though not generally a very useful one. The interesting stuff starts at $n = 1$.

These are (finite dimensional real) vector spaces. There is also a norm, and an inner (or dot) product, and a distance on each vector space $\mathbb{R}^n$, a cross product on $\mathbb{R}^3$ and a bunch of other structures with which I assume the reader is familiar and will recognize in the notation I use for them when they appear.

I haven't yet decided[1] if the elements of $\mathbb{R}^n$ should be considered row vectors or column vectors. For now, notice I've used row vectors, so maybe I should write $\mathbb{R}^n_{\text{row}}$ and

$$\mathbf{x}^T = (x_1, x_2, \ldots, x_n)^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n_{\text{column}},$$

[1] After 40+ years of indoctrination in this religion.

but for now I'm not going to do that...or not much of that. I'll also sometimes ignore the transpose altogether and write something like

$$A\mathbf{x}$$

where $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ is an m by n matrix when I technically mean

$$A\mathbf{x}^T$$

to really technically make sense or even

$$(A\mathbf{x}^T)^T$$

if I want to emphasize the technicality that I'm considering the product in $\mathbb{R}^n = \mathbb{R}^n_{\text{row}}$.

I may get more careful about this if I figure out how to do so consistently and think it's worth it.

I'll also sometimes use $\mathbf{x}$ for $(x_1, x_2, \dots, x_n)$ and sometimes $x = (x_1, x_2, \dots, x_n)$. Notice the different type faces. And sometimes I'll use them together:

Returning to sets, given $r > 0$ and $p \in \mathbb{R}^n$,

$$B_r(p) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - p| < r\}$$

is the (open) **ball** of radius r and center p in $\mathbb{R}^n$. The corresponding **closed ball** shall be denoted

$$\overline{B_r(p)},$$

and overlines will generally denote closures for sets, though overlines may also denote other things too (who knows).

Intervals: $B_r(p) \subset \mathbb{R}^1$ is $(p - r, p + r)$, and $\overline{B_r(p)} \subset \mathbb{R}$ is $[p - r, p + r]$.

For a while here[2] let's say $p \in \mathbb{R}^n$ is a given point and $r > 0$ is a given number/radius. Say $\mathbf{v} \in \mathbb{R}^n$ is also a given vector, which I'm just going to think of as a point in $\mathbb{R}^n$, but perhaps should also be thought of as an element of the tangent space $T_p\mathbb{R}^n$.

A **line** in $\mathbb{R}^n$ (through p in the direction $\mathbf{v}$) is

$$\ell(p, \mathbf{v}) = \{p + t\mathbf{v} : t \in \mathbb{R}\}.$$

[2]I would say "until further notice," but I have no intention of offering any notice of the discontinued practice.

For nonempty sets A and B in $\mathbb{R}^n$ the **distance** between A and B is always well-defined by

$$\text{dist}(A, B) = \inf\{|y - x| : x \in A,\ y \in B\}.$$

Notice $|y - x|$ here denotes the (Euclidean) norm. Note that I do not dignify the Euclidean norm by writing $\|y - x\|$, though I might use this kind of notation especially if I'm using a different norm or a norm on an infinite dimensional vector space. In any case, here is a nice result:

Theorem 2. Given $A, B \subset \mathbb{R}^n$ with $A, B \neq \phi$, there exist $p \in \overline{A}$ and $q \in \overline{B}$ with

$$|q - p| = \text{dist}(A, B).$$

This is most easily remembered and used when A and B are closed, so that $\overline{A} = A$ and $\overline{B} = B$.

The (solid) open **cylinder**[3] is given by

$$\Sigma_r(p, \mathbf{v}) = \{\mathbf{x} \in \mathbb{R}^n : \text{dist}(\mathbf{x}, \ell(p, \mathbf{v})) < r\}.$$

Note the unimportant special cases: If $\mathbf{v} = \mathbf{0}$, then $\Sigma_r(p, \mathbf{0}) = B_r(p)$ since $\ell(p, \mathbf{v}) = \{p\}$. This silliness can be avoided by requiring $\mathbf{v} \neq \mathbf{0}$. An especially nice instance of silliness avoidance is

$$\mathbf{v} \in \mathbb{S}^{n-1} = \partial B_1(\mathbf{0}) \subset \mathbb{R}^n. \tag{1.1}$$

Here of course $\partial A = \overline{A} \cap \overline{A^c}$ is the **boundary** of A. (If I've got my topology right, that's a correct definition (at least in a T-73 space).)

Exercise 1.1. In this case one can also write

$$\partial B_1(\mathbf{0}) = \overline{B_1(\mathbf{0})} \backslash B_1(\mathbf{0})$$

but it's not a correct to imagine

$$\partial A = \overline{A} \backslash A \tag{1.2}$$

for a general set A, so (1.2) doesn't give a good definition. (Give an example illustrating the badness of (1.2).

[3]. . . which maybe I should call a "bottle" or something else since a "cylinder" is usually a kind of surface.

The set in (1.1) is called the $(n-1)$ **sphere in** $\mathbb{R}^n$, and (1.1) is what some authors mean by calling $\mathbf{v}$ a **direction**. I'll probably be a little more permissive on this and sometimes refer to any nonzero vector $\mathbf{v}$ as a direction, or maybe even the zero vector, which makes the use of the term direction a little ridiculous or at least inefficient. On the other hand, I like to have at least three words for "set" including collection and class, so I suppose I do not aim for maximum efficiency.

For a second unimportant special case note that if $n = 1$ and $\mathbf{v} \neq \mathbf{0}$, then $\Sigma_r(p, \mathbf{v}) = \mathbb{R}$ (because $\ell(p, \mathbf{v}) = \mathbb{R}$). I guess this is either a good reason not to use a special notation for lines in $\mathbb{R}^1$ or to strictly embrace the silliness of $\mathbf{v} = \mathbf{0}$ in $\mathbb{R}^1$ where $\mathbb{S}^0 = \{\pm 1\}$.

A somewhat less unimportant example, which is sometimes pretty useful in the correct context arises when $n = 2$ and $\mathbf{v} \neq \mathbf{0}$. Then $\Sigma_r(p, \mathbf{v})$ is a **strip** of width $2r$ ("centered" at p and in the direction $\mathbf{v}$).

In a certain sense what I'm really aiming for here is what's sometimes called a **finite open cylinder** as follows: Given $h > 0$ and $\mathbf{v} \in \mathbb{S}^{n-1}$,

$$\Sigma_{r,h}(p, \mathbf{v}) = \{\mathbf{x} \in \mathbb{R}^n : \operatorname{dist}(\mathbf{x}, \ell(p, \mathbf{v})) < r,\ |(\mathbf{x} - p) \cdot \mathbf{v}| < h\}.$$

Such a set is said to have "height" $2h$. It can be amusing to think about $\Sigma_r(\mathbf{0}, \mathbf{e}_4) \subset \mathbb{R}^4$ to pretty comprehensively challenge the idea of calling such a set a "cylinder." But if we stick to open solid cylinders in $\mathbb{R}^n$ for $n \leq 3$, we shouldn't get into too much trouble.

1.2 Regularity classes

Given $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and an open set $U \subset \mathbb{R}^n$, a real valued function $f : U \to \mathbb{R}$ is **continuous** at $p \in U$ if for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$|f(x) - f(p)| < \epsilon \qquad \text{whenever} \qquad |x - p| < \delta.$$

The function f is **continuous on the set** U if f is continuous at each point $p \in U$. The collection of all real valued continuous functions on U is denoted by $C^0(U)$.

Exercise 1.2. Show $f : U \to \mathbb{R}$ is continuous on U if and only if $f^{-1}(V) = \{x \in U : f(x) \in V\}$ is open in U for every open subset V of $\mathbb{R}$.

Exercise 1.3. Explain why

$$\|f\|_{C^0} = \sup_{x\in U} |f(x)|$$

does **not** make $C^0(U)$ a normed vector space.

The definition of continuity at a point may also be generalized to allow a real valued function $f : A \to \mathbb{R}$ with domain any subset $A \subset \mathbb{R}^n$. Specifically, f is continuous at $p \in A$ for any $\epsilon > 0$ there exists some $\delta > 0$ for which

$$|f(x) - f(p)| < \epsilon \qquad \text{whenever} \qquad \begin{cases} |x-p| < \delta, \text{ and} \\ x \in A. \end{cases}$$

In this case one is said to be using the **subspace topology** on the set A generated by $\{U \cap A : U \text{ is open in } \mathbb{R}^n\}$. In this way one may also consider the set $C^0(A)$.

If K is a closed and bounded set, and if $K = \overline{U}$ is the closure of a bounded open set U in particular, then $C^0(K)$ is a normed vector space with norm

$$\|f\|_{C^0} = \max_{x\in K} |f(x)|.$$

In fact, in this case $C^0(K)$ is metrically complete: That is, $C^0(K)$ is a Banach space. This holds also when $\mathcal{S}$ is a compact surface[4] in $\mathbb{R}^3$: That is $C^0(\mathcal{S})$ is a Banach space when $\mathcal{S}$ is compact.

Whenever A is closed there is an alternative formulation for $C^0(A)$:

Exercise 1.4. Let $A \subset \mathbb{R}^n$ be a closed set. Show $f \in C^0(A)$ if an only if there exists an open set U and a function $g \in C^0(U)$ such that

$$g\big|_A \equiv f,$$

that is, there is an extension of f to an open set U containing A. Hint: The open set U can be taken to be all of $\mathbb{R}^n$.

The situation is less simple with derivatives. Here we restrict, at least initially, to an open set $U \subset \mathbb{R}^n$:

$$C^1(U) = \left\{ f \in C^0(U) : \frac{\partial f}{\partial x_j} \in C^0(U), j = 1, 2, \dots, n \right\}.$$

[4]For a careful definition see Chapter 2 below.

We can also then define

$$C^1(\overline{U}) = \Big\{ f \in C^0(\overline{U}) : \text{for some open set } V \text{ one has} \\ V \supset \overline{U} \text{ and } g\big|_{\overline{U}} \equiv f \text{ for some } g \in C^1(V) \Big\}.$$

Alternatively, one may consider the larger collection[5]

$$C^1(\overline{U}) = \left\{ f \in C^0(\overline{U}) \cap C^1(U) : \text{there exist } g_j \in C^0(\overline{U}) \\ \text{with } g_j\big|_U \equiv \frac{\partial f\big|_U}{\partial x_j} \text{ for } j = 1, 2, \ldots, n \right\}.$$

The second formulation does not *always* give a larger set:

Exercise 1.5. If $n = 1$ and $U = (a, b)$ is a nonempty open interval in $\mathbb{R}$, show the sets

$$\Big\{ f \in C^0[a, b] : g\big|_{[a,b]} \equiv f \text{ for some } g \in C^1(\mathbb{R}) \Big\}$$

and

$$\Big\{ f \in C^0[a, b] \cap C^1(a, b) : g\big|_{(a,b)} \equiv f' \text{ for some } g \in C^0(\overline{U}) \Big\}$$

are the same set.

To see the two definitions given for $C^1(\overline{U})$ are different *in general,* consider the following example.

Example 1. For $n = 2$, consider the union of disks given by

$$U = \bigcup_{m=1}^{\infty} [B_{1/2^{4m}}(1/2^m, 0) \cup B_{1/2^{4m}}(1/2^m + 1/2^{2m}, 0)].$$

[5]Note that when we say $f \in C^0(\overline{U}) \cap C^1(U)$ here, we technically mean $f \in C^0(\overline{U})$ and $f\big|_U \in C^1(U)$.

These open disks are disjoint with $\overline{U} = U \cup \{(0,0)\}$, so $f : \overline{U} \to \mathbb{R}$ by

$$f(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \in B_{1/2^{4m}}(1/2^m, 0) \cup \{(0,0)\} \\ & \\ \dfrac{1}{2^{2m}} & \mathbf{x} \in B_{1/2^{4m}}(1/2^m + 1/2^{2m}, 0), \end{cases} \qquad , \qquad m = 1, 2, 3, \ldots$$

gives a well-defined piecewise constant function $f \in C^0(\overline{U})$ with $Df(\mathbf{x}) \equiv (0,0)$ on U, so there also holds $f\big|_U \in C^1(U)$.

Consequently, setting $g_j(\mathbf{x}) \equiv 0$ for $j = 1, 2$ we obtain functions $g_j \in C^(\mathbb{R}^2)$ for which

$$g_j\big|_U \equiv \frac{\partial f}{\partial x_j}, \qquad j = 1, 2.$$

On the other hand, there does not exist a function $h \in C^1(V)$ defined on an open set $V \supset \overline{U}$ for which $h\big|_U \equiv f$. Were such a set V and such a function h to exist, one could take some $\epsilon > 0$ and a ball $B_\epsilon(0,0) \subset V$ so that

$$B_{1/2^{4m}}(1/2^m, 0) \cup B_{1/2^{4m}}(1/2^m + 1/2^{2m}, 0) \subset B_\epsilon(0,0)$$

for all m large enough. This would mean, on the one hand, that

$$\frac{\partial h}{\partial x_1}(0,0) = \lim_{m \nearrow \infty} \frac{\partial h}{\partial x_1}\left(\frac{1}{2^m}, 0\right) = 0. \tag{1.3}$$

On the other hand, for each m large enough there is some ξ_m with $1/2^m < \xi_m < 1/2^m + 1/2^{2m}$ such that

$$1 = 2^{2m} h(1/2^m + t/2^{2m}, 0)\Big|_{t=0}^{1} = \frac{\partial h}{\partial x_1}(\xi_m, 0).$$

In particular, this means

$$\frac{\partial h}{\partial x_1}(0,0) = \lim_{m \nearrow \infty} \frac{\partial h}{\partial x_1}(\xi_m, 0) = 1$$

which contradicts (1.3).

Exercise 1.6. Draw the domain in Example 1 above.

Exercise 1.7. Give a version of Example 1 in the case $n = 1$. This is essentially Whitney's original example from [7].

Exercise 1.8. The domain U in Example 1 is highly disconnected. Give an example showing the two versions of $C^1(\overline{U})$ can be different when the domain U is (simply) connected.

1.3 Higher dimensional domains and codomains

To make the questionable distinction between calculus and (differential) geometry, one might say the fundamental object of calculus is the function $\psi : U \to \mathbb{R}^m$ where U is an open subset of $\mathbb{R}^n$. At least I would like to consider such functions as fundamental objects from calculus at the moment and say a few things about them including fixing some associated notation.

First of all such a function has m **coordinate functions**

$$\psi = (\psi_1, \psi_2, \ldots, \psi_m).$$

Each coordinate function is a real valued function $\psi_i : U \to \mathbb{R}$ of the sort contemplated in the previous section above. In particular, these coordinate functions might fall into various regularity classes. For example, perhaps we have $\psi_i \in C^k(U)$ for $i = 1, 2, \ldots, m$. In this case I'll write $\psi \in C^k(U \to \mathbb{R}^m)$. If $k \geq 1$ then derivatives will be involved:

$$D\psi_i = \left(\frac{\partial \psi_i}{\partial x_1}, \frac{\partial \psi_i}{\partial x_2}, \ldots \frac{\partial \psi_i}{\partial x_n} \right) \in \mathbb{R}^n$$

and

$$D\psi = \left(\frac{\partial \psi_i}{\partial x_j} \right) \in \mathbb{R}^{m \times n}.$$

Arranging the partial derivatives in this way, along with playing fast and loose with the distinction between row vectors and column vectors makes it convenient to consider the linear function

$$d\psi_{\mathbf{u}} : \mathbb{R}^n \to \mathbb{R}^m \qquad \text{given by} \qquad d\psi_{\mathbf{u}}(\mathbf{v}) = D\psi(\mathbf{u})\mathbf{v}$$

which is called the **differential** of ψ. In some framework one should probably start to imagine distinct domains and codomains for the differential so that

$$d\psi_{\mathbf{u}} : T_{\mathbf{u}}\mathbb{R}^n \to T_{\psi(\mathbf{u})}\mathbb{R}^m$$

where $T_{\mathbf{u}}\mathbb{R}^n$ and $T_{\psi(\mathbf{u})}\mathbb{R}^m$ are the tangent spaces to $\mathbb{R}^n$ and $\mathbb{R}^m$ at the points $\mathbf{u}$ and $\psi(\mathbf{u})$ respectively respectively. One can also imagine writing

$$(D\psi(\mathbf{u})\mathbf{v}^T)^T,$$

or thinking of the tangent spaces as consisting of column vectors or some such, but I'm not yet convinced there is much value in such care.

It does seem like a good idea to have some idea of the derivative(s) and the diffrential of a function $\psi \in C^1(U \to \mathbb{R}^m)$, and some special cases are of particular interest.

One of those special cases is when $n = m$ where the matrix $D\psi$ is square and the possibility of an inverse arises. This is the kind of function under consideration in the results of the next section.

Another main case for us is when $X : U \to \mathbb{R}^3$ is a parameterization of a surface. If this is a somehow unfamiliar idea, then starting with the familar quadratic graphs from calculus, for example

$$\{(x_1, x_2, x_1^2 + x_2^2) : (x_1, x_2) \in \mathbb{R}^2\},$$

$$\{(x_1, x_2, x_2^2) : (x_1, x_2) \in \mathbb{R}^2\},$$

$$\{(x_1, x_2, x_1^2 - x_2^2) : (x_1, x_2) \in \mathbb{R}^2\},$$

$$\{(x_1, x_2, x_1 x_2) : (x_1, x_2) \in \mathbb{R}^2\},$$

and so forth might be a good place to start. Each of these is parameterized by what O'Neil calls a "Monge patch" $X : U \to \mathbb{R}^3$ with $X(x_1, x_2) = (x_1, x_2, f(x_1, x_2))$ for some $f \in C^\infty(\mathbb{R}^2)$. After a while one comes to realize such surfaces have an existence independent from the initial parameterization or coordinate representation with which they are first encountered. This might be the way Euclid or Apollonius thought of a triangle or a quadric curve or surface. As an independent entity—independent of coordinates—such a surface can be considered geometrically either without coordinates or with the later Cartesian violence.[6]

Clearly we begin our study here with surfaces as subsets of a fixed Euclidean coordinate system. Nevertheless one also may come to realize that the geometric properties of a surface may be considered via computations using many distinct parameterizations for the same coordinate independent geometric entity. Thus, one may wish to accomplish the determination of geometric properties using an arbitrary coordinate parameterization. This is the point of view taken in these notes.

For example if one wishes to analyze the shape of a certain physical structure like an aerofoil, then it is natural to consider some parameterization

[6]Hermann Weyl, in reference to coordinate parameterizations, famously called the introduction of coordinates an act of violence.

of the curving surface. A very clumsy approximation might be given by a finite or capped cylinder parameterized by

$$\begin{cases} X_0 : B_r(\mathbf{0}) \to \mathbb{R}^3 & \text{by} \quad X_0(u_1, u_2) = (u_1, u_2, 0) \\ X_1 : \mathbb{R} \times [0, h] \to \mathbb{R}^3 & \text{by} \quad X_1(\theta, z) = (r\cos\theta, r\sin\theta, z) \\ X_2 : B_r(\mathbf{0}) \to \mathbb{R}^3 & \text{by} \quad X_0(u_1, u_2) = (u_1, u_2, h). \end{cases}$$

One might use different parameterizations, but the utility of using some such means for calculating geometric quantities and understanding the geometric properties of such a structure should be more or less obvious. The ability to use any given parameterization is a natural next step.

Returning to surfaces familiar from calculus, consideration of the sphere

$$\partial B_r(\mathbf{0}) = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < r^2\}$$

should serve to alert the student of calculus to at least the ideas that

1. It is not possible to find a single parameterization covering all points of $\partial B_r(\mathbf{0})$.

2. The two Monge patches $X_\pm \in C^\infty(U \to \mathbb{R}^3)$ with $U = B_r(\mathbf{0}) \subset \mathbb{R}^2$ determined by

   $$f_+(x_1, x_2) = \sqrt{r^2 - x_1^2 - x_2^2} \qquad \text{and} \qquad f_-(x_1, x_2) = -\sqrt{r^2 - x_1^2 - x_2^2},$$

   in addition to not completely covering the sphere are somewhat deficient with respect to differentiation because, for example,

   $$\lim_{|(x_1,x_2)|\to r} |Df_\pm(x_1, x_2)| = \lim_{|(x_1,x_2)|\to r} \sqrt{\left(\frac{\partial f_\pm}{\partial x_1}(x_1, x_2)\right)^2 + \left(\frac{\partial f_\pm}{\partial x_2}(x_1, x_2)\right)^2}$$
   $$= +\infty.$$

Again, hopefully the example of the sphere serves to motivate the consideration of a surface $\mathcal{S}$ like $\partial B_r(\mathbf{0}) \subset \mathbb{R}^3$ or pieces of such a surface $\mathcal{S}$ in terms of many different, and even **all** different, coordinate parameterizations $X : U \to \mathbb{R}^3$.

As for geometry, it may be helpful to to consider the subject from a start in asking questions like:

1. What different coordinate parameterizations can be found for a given surface?

2. Given a coordinate parameterization for a surface, in what way can that surface be identified or classified? Is the surface a sphere? Is the surface some quadric surface of analytic geometry? Is the surface something else?

And if this is the beginning, then the end might be summed up as "curvature" with the question: How does one compute the curvature of a surface and what does the computation say about the geometry? The end might be pushed a little further with the notion of energy with Willmore energy, Dirichlet energy, and even area (energy) giving examples.

Focusing on the "stretch" between the beginning consisting of parameterizations of surfaces in one form or another and the end of curvature/energy for surfaces, there are a number of intermediate steps very worthy of consideration. There is for example the calculation of lengths of curves on surfaces. This has also associated with it its own peculiar energy of length and the local minimizers called geodesics. More generally, there are various foliations of surfaces by curves and also the singular line fields—especially integrable ones considered by Heinz Hopf. One may—and really must—also consider tangent spaces and fields of tangent vectors and some associated forms. One can run the risk of bundleing and tensoring one's self into oblivion at this point and never thinking of curvature again. Fortunately with due respect to integration on surfaces, there is hope—with some self-restraint and moderation—to reach the end—or at least some appreciation for the end of curvature.

Finally it may be suggested that the consideration of all concepts in one lower dimension can be useful for discerning the context and motivations involved in the study of surfaces in $\mathbb{R}^3$ presented below. That is to say one may consider a subset $C \subset \mathbb{R}^2$ obtained as a union of appropriate regular parameterizations $\alpha \in C^k(I \to \mathbb{R}^2)$ with I an open interval in $\mathbb{R}$. If the specialization is done correctly, one should obtain a serviceable framework for understanding the geometry of planar curves.

1.4 Inverse and implicit functions

Theorem 3. (inverse function theorem) Given an open set $U \subset \mathbb{R}^n$ and a function $\psi \in C^k(U \to \mathbb{R}^n)$ for which there is a point $\mathbf{q} \in U$ with

$$\det D\psi(\mathbf{q}) \neq 0,$$

there exists on open set U_0 with $\mathbf{q} \in U_0 \subset U$ such that

$$\psi\big|_{U_0} : U_0 \to \psi(U_0)$$

is a C^k diffeomorphism.

It is often convenient to determine a function implicitly using the inverse function theorem. The simplest example might be the following:

Theorem 4. Given an open set $U \subset \mathbb{R}^2$ and a function $f \in C^k(U)$ with a point $\mathbf{q} = (q_1, q_2) \in U$ such that

$$\frac{\partial f}{\partial y}(\mathbf{q}) \neq 0,$$

there exists some $\epsilon > 0$ and a function $w \in C^k(q_1 - \epsilon, q_1 + \epsilon)$ for which

$$\begin{cases} f(x, w(x)) = f(\mathbf{q}) \text{ and} \\ w(q_1) = q_2. \end{cases} \tag{1.4}$$

The conditions (1.4) determine the function w uniqueley in any open subinterval of $(q_1 - \epsilon, q_1 + \epsilon)$ containing q_1.

Proof: Let $\psi : U \to \mathbb{R}^2$ be defined by

$$\psi(x, y) = (x, f(x, y)).$$

Then $\psi(\mathbf{q}) = (q_1, f(\mathbf{q}))$ and

$$D\psi = \begin{pmatrix} 1 & 0 \\ f_x & f_y \end{pmatrix},$$

so

$$\det D\psi(\mathbf{q}) = f_y(\mathbf{q}) = \frac{\partial f}{\partial y}(\mathbf{q}) \neq 0.$$

The inverse function theorem applies, and we may denote the component functions of

$$\left(\psi\Big|_{U_0}\right)^{-1} : \psi(U_0) \to U_0$$

by w_1 and w_2. Since $\psi(U_0)$ is open and contains $(q_1, f(\mathbf{q}))$, we may take some $\epsilon > 0$ so that the ball $B_\epsilon(q_1, f(\mathbf{q}))$ satisfies $B_\epsilon(q_1, f(\mathbf{q})) \subset \psi(U_0)$. Finally then we define $w \in C^k(q_1 - \epsilon, q_1 + \epsilon)$ by

$$w(\xi) = w_2(\xi, f(\mathbf{q})).$$

Because ψ^{-1} is the inverse of ψ, we know there holds

$$(w_1(x, f(x,y)), w_2(x, f(x,y)) = (x,y) \qquad \text{for} \qquad (x,y) \in U_0.$$

This implies in particular that $w_1(x, f(x,y)) = x$. Also, specializing this identity to $(x,y) = \mathbf{q}$ we see from the second component

$$w(q_1) = w_1(q_1, f(\mathbf{q})) = q_2.$$

This is the second condition in (1.4).

On the other hand, because ψ is the inverse of ψ^{-1}, we have

$$(w_1(\xi,\eta), f(w_1(\xi,\eta), w_2(\xi,\eta))) = (\xi,\eta) \qquad \text{for} \qquad (\xi,\eta) \in B_\epsilon(q_1, f(\mathbf{q})).$$

Noting that the first component identity gives $w_1(\xi,\eta) = \xi$ independent of η, we may specialize η to $f(\mathbf{q})$ in the second component to see

$$f(\xi, w(\xi)) = f(\mathbf{q}) \qquad \text{for} \qquad q_1 - \epsilon < \xi < q_1 + \epsilon.$$

This is the first condition of (1.4).

We have found a function $w \in C^k(q_1 - \epsilon, q_1 + \epsilon)$ satisfying the existence assertion of the theorem, and it remains to establish uniqueness.

Assume $\tilde{w} \in C^k(a,b)$ for some $(a,b) \subset (q_1 - \epsilon, q_1 + \epsilon)$ with $q_1 \in (a,b)$ satisfies

$$\begin{cases} f(x, \tilde{w}(x)) = f(\mathbf{q}) \text{ and} \\ \tilde{w}(q_1) = q_2. \end{cases} \tag{1.5}$$

Then

$$\begin{aligned} \psi(x, \tilde{w}(x)) &= (x, f(x, \tilde{w}(x))) \\ &= (x, f(\mathbf{q})) \\ &= (x, f(x, w(x))) \\ &= \psi(x, w(x)). \end{aligned} \tag{1.6}$$

By continuity, there is some $\delta > 0$ for which $(x, \tilde{w}(x)) \in U_0$ at least for $q_1 - \delta < x < q_1 + \delta$. Consequently, we may apply $\psi\big|_{U_0}^{-1}$ to (1.6) to obtain

$$(x, \tilde{w}(x)) = (x, w(x)) \qquad \text{for} \qquad q_1 - \delta < x < q_1 + \delta. \tag{1.7}$$

Set

$$\begin{cases} a_0 = \inf\{\alpha \in (a, q_1) : \tilde{w}(x) = w(x),\ \alpha \le x \le q_1\} \text{ and} \\ b_0 = \inf\{\beta \in (q_1, b) : \tilde{w}(x) = w(x),\ q_1 \le x \le \beta\} \text{ and.} \end{cases}$$

Assuming $a_0 > a$ we observe

$$\tilde{w}(a_0) = \lim_{x \searrow a_0} \tilde{w}(x) = w(a_0) \qquad \text{and} \qquad (a_0, w(a_0)) \in U_0.$$

Accordingly there is some $\delta > 0$ for which the relation (1.6) is still valid for $a_0 - \delta < x \le a_0$. Applying $\psi\big|_{U_0}^{-1}$ as before we find the identity of (1.7) also extends to $a_0 - \delta < x < a_0$, and this contradicts our assumption $a_0 > a$. We conclude $a_0 = a$ and $\tilde{w}(x) \equiv w(x)$ for $a < x < b_0$.

We also obtain a contradiction unless $b_0 = b$, so the uniqueness of the implicit function w is established. $\square$

Exercise 1.9. State and prove a theorem with the hypotheses of Theorem 4 giving an implicit function $w \in C^k(a, b)$ on a **maximal** interval $(a, b) \subset \mathbb{R}$.

Chapter 2

Surfaces

2.1 Preliminary definitions of a surface

Definition 1. (specific/restrictive definition after O'Neil and Spivak) Given $\mathcal{S} \subset \mathbb{R}^3$ and $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, the set $\mathcal{S}$ a C^k **embedded surface** if for each $p \in \mathcal{S}$ there is some $\epsilon > 0$ and a function (parameterization)

$$X \in C^k(\mathbb{R}^2 \to \mathbb{R}^3)$$

such that the following hold:

(0) $X(0, 0) = p$.

(1) $X : \mathbb{R}^2 \to B_\epsilon(p) \cap \mathcal{S}$ is a homeomorphism.[1]

(2) For each $\mathbf{u} \in \mathbb{R}^2$ the differential map $dX_\mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

The last condition is equivalent to the condition that the derivative matrix

$$DX(\mathbf{u}) = \begin{pmatrix} \dfrac{\partial X_1}{\partial u_1}(\mathbf{u}) & \dfrac{\partial X_1}{\partial u_2}(\mathbf{u}) \\ \dfrac{\partial X_2}{\partial u_1}(\mathbf{u}) & \dfrac{\partial X_2}{\partial u_2}(\mathbf{u}) \\ \dfrac{\partial X_3}{\partial u_1}(\mathbf{u}) & \dfrac{\partial X_3}{\partial u_2}(\mathbf{u}) \end{pmatrix}$$

[1] For continuity here we take $\mathcal{S}$ as a topological subspace of $\mathbb{R}^3$ with the (subspace) topology $\{V \cap \mathcal{S} : V \text{ is open in } \mathbb{R}^3\}$, and $\xi : B_\epsilon(p) \cap \mathcal{S} \to \mathbb{R}$ is the associated chart which is the inverse of X.

has rank 2 where $X = (X_1, X_2, X_3)$. Another equivalent formulation is that the vectors

$$X_{u_1}(\mathbf{u}) = \left(\frac{\partial X_1}{\partial u_1}(\mathbf{u}), \frac{\partial X_2}{\partial u_1}(\mathbf{u}), \frac{\partial X_3}{\partial u_1}(\mathbf{u}) \right) \qquad \text{and}$$

$$X_{u_2}(\mathbf{u}) = \left(\frac{\partial X_1}{\partial u_2}(\mathbf{u}), \frac{\partial X_2}{\partial u_2}(\mathbf{u}), \frac{\partial X_3}{\partial u_2}(\mathbf{u}) \right)$$

are linearly independent in $\mathbb{R}^3$ or have nonzero cross product. We will come back to this cross product when we talk about normals to the surface $\mathcal{S}$.

Definition 2. (non-specific/permissive definition after Do Carmo and Gallot) Given $\mathcal{S} \subset \mathbb{R}^3$ and $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, the set $\mathcal{S}$ a C^k **embedded surface** if for each $p \in \mathcal{S}$ there is some open $V \subset \mathbb{R}^3$ with $p \in V$, some open set $U \subset \mathbb{R}^2$, and a homeomorphism

$$X \in C^k(\mathbb{R}^2 \to \mathbb{R}^3)$$

such that $dX_{\mathbf{u}} : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one for each $\mathbf{u} \in \mathbb{R}^2$.

Clearly a C^k embedded surface according to Definition 1 is a C^k embedded surface according to Defnition 2. Just take $V = B_\epsilon(p)$ and $U = \mathbb{R}^2$. It needs to be checked that a C^k embedded surface according to Defnition 2 is a C^k embedded surface according to Definition 1.

A parameterization $X : \mathbb{R}^2 \to \mathbb{R}^3$ satisfying the requirements of Definition 1 may be referred to as a **specific initial parameterization**. A parameterization $X : U \to \mathbb{R}^3$ satisfying the requirements of Definition 2 may be called a **permissive initial parameterization** or **non-normalized parameterization** or just a **parameterization**. Thus a specific initial parameterization is a parameterization.

2.1.1 Regular mappings and local homeomorphism(s)

Certainly not every parameterization is a specific initial parameterization. Sometimes the term **regular parameterization** or **regular parameter map** is used to refer to any function $X \in C^k(U \to \mathbb{R}^3)$ where U is an open set in $\mathbb{R}^2$ and the regularity condition

$$X_{u_1} \times X_{u_2} \neq \mathbf{0}$$

holds. This is the approach taken by Heinz Hopf (according to the notes of John Gray) in introducing/summarizing "differential geometry in the small." Note there is no assumption that such a map is a local homeomorphism. Indeed, it might be very natural to contemplate parameterizing a set $\mathcal{S}$ by such a mapping which is not one-to-one. Such a map may be considered locally one-to-one, and for many purposes gives an adequate framework for geometric considerations related to surfaces. Certainly Hopf intended to allow this (see Exercise 2.2 part **(b)** below).

Exercise 2.1. Given a C^k regular parameter map $X : U \to \mathbb{R}^3$ defined on an open set $U \subset \mathbb{R}^2$ and a point $\mathbf{u} \in U$, show there is some open subset $W \subset U$ for which $\mathbf{u} \in W$ and the function

$$X\big|_W : W \to X(W)$$

is one-to-one.

One must be a little careful if one wishes to claim such a function is a **local homeomorphism**. the problem is that $X(W)$ is not necessarily open in the subspace topology of $X(U)$.

Exercise 2.2. Let $X : U \to \mathbb{R}^3$ be a C^k regular parameter map defined on an open set $U \subset \mathbb{R}^2$ and a point $\mathbf{u} \in U$ as in Exercise 2.1.

(a) Show there is an open set $W \subset U$ for which $\mathbf{u} \in W$ and the function Show $X : W \to X(W)$ is a homeomorphism with respect to the subspace topology on $X(W)$ as a subset of either $\mathbb{R}^3$ or $X(U)$.

The usual definition of local homeomorphism, however, requires also $X(W)$ is open in $X(U)$ so the inverse

$$\left(X\big|_W\right)^{-1} : X(W) \to W$$

has domain an open set in $X(U)$.

(b) Show $X : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ by $X(\theta, z) = (\cos\theta, \sin\theta, z)$ with $U = \mathbb{R} \times \mathbb{R}$ is an example in which for each $X(W)$ is open in in $X(U)$ so $X : U \to X(U)$ is a local homeomorphism and

$$\left(X\big|_W\right)^{-1} : X(W) \to W$$

is a continuous function with domain an open subset of $X(U)$.

(c) Give an example where $X : U \to X(U)$ is not a local homeomorphism.

You be the judge: Here is a direct translation of the proof from [3] with some boldface added by me for emphasis.

> **Theorem** The map $X : U \to \mathbb{R}^3$ is a local homeomorphism, i.e., the map gives a homeomorphism between a neighborhood of each point and the image of the neighborhood under the map.
>
> Proof: We may assume
>
> $$\det \begin{pmatrix} \dfrac{\partial X_1}{\partial u_1}(\mathbf{u}) & \dfrac{\partial X_2}{\partial u_1}(\mathbf{u}) \\ \\ \dfrac{\partial X_1}{\partial u_2}(\mathbf{u}) & \dfrac{\partial X_2}{\partial u_2}(\mathbf{u}) \end{pmatrix} \neq 0.$$
>
> But then the projection of **the surface** into the x_1, x_2-plane is a local homeomorphism of the u_1, u_v-plane into the x_1, x_2-plane (since the Jacobian of this map is not zero). Therefore the map **into the surface** is locally one-to-one and **open**, and hence a local homeomorphism.

To be fair, it may be that Gray (or Heinz Hopf) was operating under a different definition of **local homeomorphism**. Perhaps the bigger question is: What is meant here by **the surface**? Earlier the surface is said to be the map $X : U \to \mathbb{R}^3$ itself, but here "the surface" seems to be clearly used to refer to a set. Is the set $X(U)$ the surface or is $X(W)$ the surface for some open set $W \subset U$?

Notice the formulation of **local homeomorphism** in the statement of the theorem and contrast this with the very similar phrasing of Munkres [4] page 334 (again translated into our context): $\ldots X : U \to X(U)$ is a local homeomorphism, i.e., each point $\mathbf{u}$ of U has a neighborhood that is mapped homeomorphically by X into **an open subset of** $X(U)$.

I have added Appendix D if you are having trouble seeing the somewhat subtle point under consideration above.

2.1.2 Overlaps

Conspicuously missing from these definitions is any discussion of the familiar "overlaps" of manifold theory. In either definition we may refer to the maps

$$X : U \to V \cap \mathcal{S}$$

as **parameterizations** and the inverses

$$\xi : V \cap S \to U$$

as **charts** or **coordinate functions** or just coordinates.

Definition 3. An **initial covering collection**

$$\mathcal{C}_0 = \{X_\alpha\}_{\alpha \in \Gamma}$$

is any collection of parameterizations $X_\alpha : U_\alpha \to V_\alpha \cap \mathcal{S}$ satisfying the requirements of the definition in question and for which

$$\mathcal{S} = \bigcup_{\alpha \in \Gamma} X_\alpha(U_\alpha).$$

The covering condition simplifies to

$$\mathcal{S} = \bigcup_{\alpha \in \Gamma} X_\alpha(\mathbb{R}^2)$$

in the case of Definition 1 with $X_\alpha : \mathbb{R}^2 \to B_{\epsilon_\alpha}(p_\alpha) \cap \mathcal{S}$ for some numbers ϵ_α and points $p_\alpha \in \mathcal{S}$ for $\alpha \in \Gamma$. Notice however that even in this case one cannot expect to index the initial covering by the points $\{p_\alpha\}_{\alpha \in \Gamma}$; see Appendix E.

Definition 4. Any collection $\mathcal{C} = \{X_\alpha\}_{\alpha \in \Gamma}$ of parameterizations $X_\alpha : U_\alpha \to V_\alpha \cap \mathcal{S}$ satisfing the requirements

1. $X_\alpha : U_\alpha \to V_\alpha \cap \mathcal{S}$ is a homeomorphism, and

2. $(dX_\alpha)_{\mathbf{u}} : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one for each $\mathbf{u} \in U_\alpha$

is said to be a **covering collection**.

The corresponding collections of charts

$$\mathcal{A}_0 = \{\xi_\alpha : X_\alpha \in \mathcal{C}_0\}, \text{ and}$$
$$\mathcal{A} = \{\xi_\alpha : X_\alpha \in \mathcal{C}\}$$

are called an **initial atlas** and an **atlas** respectively.

The largest possible covering collection $\mathcal{C}_{\mathcal{M}}$ is unique and is called the **maximal covering collection**. The corresponding atlas $\mathcal{A}_{\mathcal{M}}$ is called the **maximal atlas**.

It is clear that the parameterizations and charts associated with a point $p \in \mathcal{S}$ where $\mathcal{S}$ is a C^k embedded surface are in no sense unique in general. In particular, there will be a multitude of "overlapping" parameterizations and corresponding "overlapping" charts in any covering collection. That is, there will be parameterizations $X, \tilde{X} \in \mathcal{C}$ with $X : U \to \mathbb{R}^3$, $\tilde{X} : \tilde{U} \to \mathbb{R}^3$, and

$$W = X(U) \cap \tilde{X}(\tilde{U}) \neq \phi.$$

Let W denote such an intersection for the moment.

Theorem 5. Given overlapping parameterizations X and $\tilde{X}$ for a C^k embedded surface $\mathcal{S}$ as just described, the functions

$$\tilde{\xi} \circ X\Big|_{\xi(W)} : \xi(W) \to \xi(\tilde{W})$$

and

$$\xi \circ \tilde{X}\Big|_{\tilde{\xi}(W)} : \tilde{\xi}(W) \to \xi(W)$$

are C^k diffeomorphisms (and inverses of each other).

2.1.3 Tangent space and tangent plane

It is convenient to note that according to the requirement[2]

$$dX_{\mathbf{u}} : \mathbb{R}^2 \to \mathbb{R}^3 \qquad \text{is one-to-one}$$

the set $\{X_{u_1}(\mathbf{u}), X_{u_2}(\mathbf{u})\}$ containing the vectors

$$X_{u_1} = \frac{\partial X}{\partial u_1} = \left(\frac{\partial X_1}{\partial u_1}, \frac{\partial X_2}{\partial u_1}, \frac{\partial X}{\partial u_2} \right) \quad \text{and} \quad X_{u_2} = \frac{\partial X}{\partial u_2} = \left(\frac{\partial X_1}{\partial u_2}, \frac{\partial X_2}{\partial u_2}, \frac{\partial X}{\partial u_2} \right)$$

is a linearly indepdendent set in $\mathbb{R}^3$ spanning a two dimensional algebraic subspace $T_{X(\mathbf{u})}\mathcal{S}$ of $\mathbb{R}^3$. In the particular case when $X(\mathbf{u}) = p$, which is a special case referred to very often, we write $T_p\mathcal{S}$. In any case, this two

[2]This injectivity condition is sometimes said to make the map $X : U \to \mathbb{R}^3$ "regular," "nondegenerate," or "immersive."

dimensional vector subspace is the **(algebraic) tangent space** of $\mathcal{S}$ at $X(\mathbf{u})$. Very often also it will be convenient to use the expressions

$$X_{u_1} \qquad \text{and} \qquad X_{u_2}$$

ambiguously to refer either to $X_{u_1}(\mathbf{u})$ and $X_{u_2}(\mathbf{u})$ for any $\mathbf{u} \in U$ which is reasonably proper, or to refer to the **coordinate basis vectors at** p which might also be expressed as $(X_{u_1})_p$ and $(X_{u_2})_p$ when $X(\mathbf{u}) = p$. This latter usage might be a little confusing at times.

There is also an affine tangent plane at p (and more generally at each point $X(\mathbf{u})$ given by

$$\{p + a_1 X_{u_1} + a_2 X_{u_2} : a_1, a_2 \in \mathbb{R}\} \subset \mathbb{R}^3.$$

This set is not always a subspace of $\mathbb{R}^3$ and shouldn't really be thought of as one. It can and should be thought of as another surface.

Exercise 2.3. Show the affine tangent plane is a surface according to the restrictive Definition 1.

I haven't thought of a good notation for the affine tangent space yet. Perhaps $\mathcal{S}_p$ is a good candidate.

The set $\{X_{u_1}, X_{u_2}\} = \{X_{u_1}(\mathbf{u}), X_{u_2}(\mathbf{u})\}$ is a basis for

$$T_p\mathcal{S} = \{a_1 X_{u_1} + a_2 X_{u_2} : a_1, a_2 \in \mathbb{R}\} \subset \mathbb{R}^3.$$

2.1.4 A first look at normals

Recall the tangent vectors can be used to define a local normal $X_{u_1} \times X_{u_2}$ to the affine tangent plane. Furthermore it is quite nice to consider a local **unit normal field** on $X(U)$ given by

$$N = \frac{X_{u_1} \times X_{u_2}}{|X_{u_1} \times X_{u_2}|}.$$

The reverse normal field is given on the same (portion of) surface by

$$\frac{\tilde{X}_{u_1} \times \tilde{X}_{u_2}}{|\tilde{X}_{u_1} \times \tilde{X}_{u_2}|}$$

where $\tilde{X} : \tilde{U} \to \mathbb{R}^3$ by $\tilde{X}(u_1, u_2) = X(u_2, u_1)$ and

$$\tilde{U} = \{(u_2, u_1) : \mathbf{u} = (u_1, u_2) \in U\}.$$

It may not be possible to have a continuous unit normal field on the entire surface

$$\mathcal{S} = \bigcup_{\alpha \in \Gamma} X_\alpha(U_\alpha).$$

The Möbius strip should convince you of this. But if there is such a continuous unit normal field, the surface is said to be **orientable**.

It will probably be useful to distinguish the local unit normal at p by writing N_p instead of just N for the local unit normal at $X(\mathbf{u})$.

2.2 A first look at local graphs

Theorem 6. *Given an orthonormal basis $\{E_1, E_2\}$ for $T_p\mathcal{S}$ with $E_1 \times E_2 = N_p$, for example,*

$$E_1 = \frac{X_{u_1}}{|X_{u_1}|}, \qquad E_2 = \frac{X_{u_2} - (X_{u_2} \cdot E_1)E_1}{|X_{u_2} - (X_{u_2} \cdot E_1)E_1|},$$

the surface $\mathcal{S}$ is **locally a graph over the affine tangent plane** *at p, i.e., there are some positive numbers δ and h, a function $f \in C^k(B_\delta(\mathbf{0}) \to \mathbb{R})$ and a graph*

$$\mathcal{G} = \{(x_1, x_2, f(x_1, x_2)) : (x_1, x_2) \in B_\delta(\mathbf{0})\}$$

such that

$$\mathcal{S} \cap \Sigma_{\delta,h}(p, N_p) = \{p + x_1E_1 + x_2E_2 + f(x_1, x_2)N_p : (x_1, x_2) \in B_\delta(\mathbf{0})\}.$$

In particular $\tilde{X}(x_1, x_2) = p + x_1E_1 + x_2E_2 + f(x_1, x_2)N_p$ defines a local parameterization for $\mathcal{S}$ at p with $\tilde{X}(0, 0) = p$ and $\tilde{X}_{x_j}(\mathbf{0}) = E_j$ for $j = 1, 2$.

Proof: We start with a local parameterization $X : U \to V \cap \mathcal{S}$. By translating U if necessary, we can assume $\mathbf{0} = (0, 0) \in U$ and $X(\mathbf{0}) = p$. Consider $\psi : U \to \mathbb{R}^2$ by

$$\psi(u_1, u_2) = ((X - p) \cdot E_1, (X - p) \cdot E_2).$$

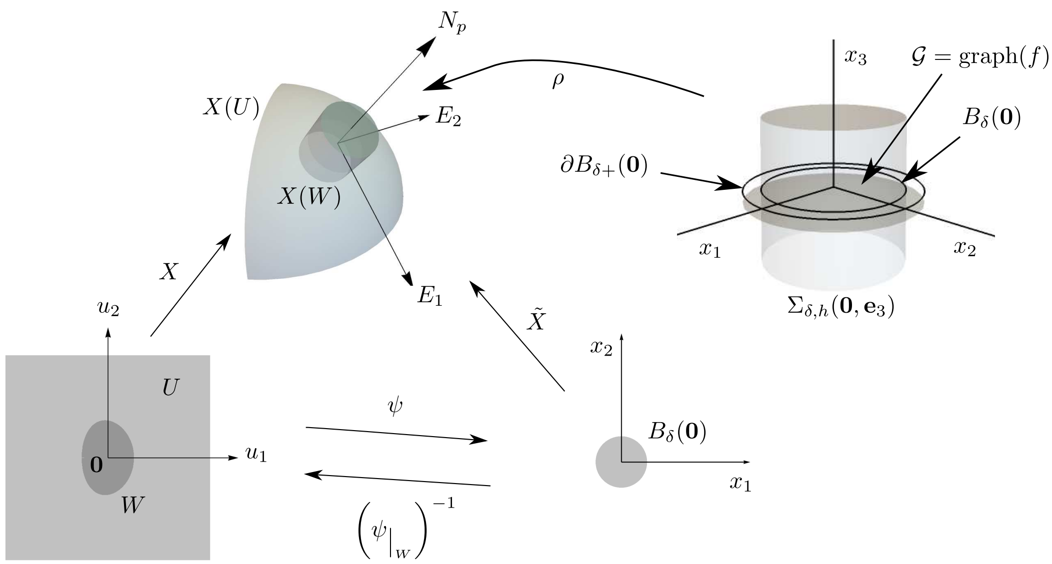

Figure 2.1: The pillbox graph theorem: Any C^k embedded surface $\mathcal{S}$ is locally the graph of a C^k function defined on a disk in the affine tangent plane $\mathcal{S}_p$ which differs from the standard graph $\mathcal{G}$ by a rigid motion.

Then

$$D\psi = \begin{pmatrix} X_{u_1} \cdot E_1 & X_{u_2} \cdot E_1 \\ X_{u_1} \cdot E_2 & X_{u_2} \cdot E_2 \end{pmatrix}.$$

Therefore, $\det D\psi = (X_{u_1} \cdot E_1)(X_{u_2} \cdot E_2) - (X_{u_2} \cdot E_1)(X_{u_1} \cdot E_2)$.

On the other hand,

$$X_{u_1} = (X_{u_1} \cdot E_1)E_1 + (X_{u_1} \cdot E_2)E_2 + (X_{u_1} \cdot N_p)N_p \quad \text{and}$$
$$X_{u_2} = (X_{u_2} \cdot E_1)E_1 + (X_{u_2} \cdot E_2)E_2 + (X_{u_2} \cdot N_p)N_p.$$

Assuming $X(\mathbf{0}) = p$ and specializing these expansions to evaluation at $\mathbf{u} = \mathbf{0}$, we have

$$X_{u_1}(\mathbf{0}) = (X_{u_1}(\mathbf{0}) \cdot E_1)E_1 + (X_{u_1}(\mathbf{0}) \cdot E_2)E_2 \quad \text{and}$$
$$X_{u_2}(\mathbf{0}) = (X_{u_2}(\mathbf{0}) \cdot E_1)E_1 + (X_{u_2}(\mathbf{0}) \cdot E_2)E_2.$$

Thus,

$$\begin{aligned}\mathbf{0} &\neq X_{u_1}(\mathbf{0}) \times X_{u_2}(\mathbf{0}) \\ &= (X_{u_1}(\mathbf{0}) \cdot E_1)(X_{u_2}(\mathbf{0}) \cdot E_2)E_1 \times E_2 + (X_{u_1}(\mathbf{0}) \cdot E_2)(X_{u_2}(\mathbf{0}) \cdot E_1)E_2 \times E_1 \\ &= \det D\psi(\mathbf{0})\ N_p.\end{aligned}$$

Notice also that $\psi(\mathbf{0}) = \mathbf{0}$. By the inverse function theorem, there is some $\delta > 0$ for which

$$\psi\big|_W : W \to B_\delta(\mathbf{0})$$

where $W = \psi^{-1}(B_\delta(\mathbf{0}))$ has a well-defined inverse

$$\left(\psi\big|_W\right)^{-1} : B_\delta(\mathbf{0}) \to W$$

with some coordinate functions $w_1, w_2 \in C^k(B_\delta(\mathbf{0}))$.

The relation

$$\psi\big|_W \circ \left(\psi\big|_W\right)^{-1} = \mathrm{id}_{B_\delta(\mathbf{0})}$$

tells us that for each $(x_1, x_2) \in B_\delta(\mathbf{0})$ there holds

$$((X(w_1, w_2) - p) \cdot E_1, (X(w_1, w_2) - p) \cdot E_2) = (x_1, x_2). \tag{2.1}$$

The inverse relation

$$\left(\psi\big|_W\right)^{-1} \circ \psi\big|_W = \mathrm{id}_W$$

yields

$$w_1((X(u_1, u_2) - p) \cdot E_1, (X(u_1, u_2) - p) \cdot E_2) = u_1 \quad \text{and}$$

$$w_2((X(u_1, u_2) - p) \cdot E_1, (X(u_1, u_2) - p) \cdot E_2) = u_2$$

for all $\mathbf{u} = (u_1, u_2) \in W = \psi^{-1}(B_\delta(\mathbf{0}))$. This second inverse relation strikes me initially as the more useful looking one of the two, but I think it turns out we only use the first one (2.1) below.

Define $f : B_\delta(\mathbf{0}) \to \mathbb{R}$ by $f(x_1, x_1) = (X(w_1, w_2) - p) \cdot N_p$. Notice that $f \in C^k(B_\delta(\mathbf{0}))$. Thus,

$$\mathcal{G} = \{(x_1, x_2, f(x_1, x_2)) : (x_1, x_2) \in B_\delta(\mathbf{0})\}$$

is the graph of a well-defined C^k function as stated in the theorem.

Finally, since $\{E_1, E_2, N_p\}$ is an orthonormal basis for $\mathbb{R}^3$ we can write for each $\mathbf{u} = (u_1, u_2) \in W$

$$X(\mathbf{u}) = p + [(X(\mathbf{u}) - p)\cdot E_1]E_1 + [(X(\mathbf{u}) - p)\cdot E_2]E_2 + [(X(\mathbf{u}) - p)\cdot N_p]N_p.$$

Substituting into this relation $\mathbf{u} = (w_1, w_2) = (w_1(x_1, x_2), w_2(x_1, x_2))$ and taking account of the first inverse relations (2.1) and the definition of f, we have for $(x_1, x_2) \in B_\delta(\mathbf{0})$

$$\tilde{X} = p + x_1E_1 + x_2E_2 + f(x_1, x_2)N_p$$

with

$$\tilde{X} = \tilde{X}(x_1, x_2) = X(w_1, w_1) = X \circ \left(\psi\big|_W\right)^{-1}.$$

Note: $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ by $\rho(x_1, x_2, x_3) = p + x_1E_1 + x_2E_2 + x_3N_p$ defines a rigid motion with linear part $A\mathbf{x} = x_1E_1 + x_2E_2 + x_3N_p$.

It remains to show the function $\tilde{X} \in C^k(B_\delta(\mathbf{0}) \to \mathbb{R}^3)$ is or can be taken to be an open map into $\mathcal{S}$ giving a homeomorphism onto its image with respect to the particular open set $\Sigma_{\delta,h}(p, N_0)$ as stated in the theorem. In order to establish this last point, we may need to take δ somewhat smaller. We will also establish something additional.

Fix $\delta = \delta_0 > 0$ so that the application of the inverse function theorem and the assertions above hold. In particular, rename the corresponding fixed graph $\mathcal{G}$ above as

$$\mathcal{G}_0 = \{(x_1, x_2, f(x_1, x_2)) : (x_1, x_2) \in B_{\delta_0}(\mathbf{0})\}.$$

Note first that $f(\mathbf{0}) = 0$ and

$$Df = \left(\left(X_{u_1}\frac{\partial w_1}{\partial x_1} + X_{u_2}\frac{\partial w_1}{\partial x_1}\right)\cdot N_0, \left(X_{u_1}\frac{\partial w_1}{\partial x_2} + X_{u_2}\frac{\partial w_1}{\partial x_2}\right)\cdot N_0\right).$$

In particular $Df(\mathbf{0}) = (0, 0)$. This means the first order Taylor expansion of f at $(x_1, x_2) = \mathbf{0}$ is constant zero. By Taylor's formula there are real valued remainder functions $R_1 = R_1(x_1, x_2)$ and $R_2 = R_2(x_1, x_2)$ for which

$$f(x_1, x_2) = R_1(x_1, x_2)x_1 + R_2(x_1, x_2)x_2$$

and

$$\lim_{|x|\to 0} R_j(x) = 0 \qquad \text{for} \qquad j = 1, 2.$$

In particular, given any ratio $\mu > 0$, it is possible to take $\delta > 0$ small enough with $\delta < \delta_0$ in particular so that

$$|f(x_1, x_2)| < \mu\delta \qquad \text{for} \qquad |(x_1, x_2)| < \delta < \delta_0.$$

This means that by taking δ small enough we always know

$$\mathcal{G} \subset \Sigma_{\delta,h}(\mathbf{0}, \mathbf{e}_3)$$

for every $h > \mu\delta$. This means the image $\rho(\mathcal{G}) \subset \Sigma_{\delta,h}(p, N_p)$ where h can be taken as an arbitrarily small multiple of δ as long as we take δ small enough.

On the other hand, we know there is an open set $V_0 = V_0(\delta_0) \subset \mathbb{R}^3$ for which $X(W) = \mathcal{S} \cap V_0$. We can then take $\delta < \delta_0$ small enough and any $h > \mu\delta$ satisfying

$$\Sigma_{\delta,h}(p, N_p) \subset V_0.$$

Then

$$\left(X\Big|_W \right)^{-1} (\Sigma_{\delta,h}(p, N_p)) \subset W.$$

In this way, we know exactly what $\mathcal{S}$ looks like inside $\Sigma_{\delta,h}(p, N_p)$. The set $\mathcal{S} \cap \Sigma_{\delta,h}(p, N_p)$ is given by the rigid motion ρ applied to the graph

$$\begin{aligned} \mathcal{G} = \{(x_1, x_2, f(x_1, x_2)) : (x_1, x_2) \in B_\delta(\mathbf{0})\} \\ \subset \mathcal{G}_0 = \{(x_1, x_2, f(x_1, x_2)) : (x_1, x_2) \in B_{\delta_0}(\mathbf{0})\}, \end{aligned}$$

and the open codomain assertion

$$\mathcal{S} \cap \Sigma_{\delta,h}(p, N_p) = \{p + x_1 E_1 + x_2 E_2 + f(x_1, x_2) N_p : (x_1, x_2) \in B_\delta(\mathbf{0})\}$$

in the statement of the theorem holds. The enhanced assertion we have obtained which is not stated in the theorem is that the ratio h/δ of the height of the finite cylindrical neighborhood may be taken smaller than any given positive number μ. $\square$

Perhaps a good name for this result is the "pillbox graph" theorem or just the pillbox theorem. It gives some control both over the shape of the parameter domain for a local parameterization $\tilde{X} : B_\delta(\mathbf{0}) \to \mathcal{S}$, which can be taken to be a disk, and over the open set $\tilde{V} = \Sigma_{\delta,h}(p, N_p)$ in $\mathbb{R}^3$, which is a matching finite cylindrical open set, or a pillbox. We also have $\tilde{X}(\mathbf{0}) = p$ here.

Exercise 2.4. The illustration in Figure 2.1 is built using explicit formulas for a local parameterization of $\mathbb{S}^2$ using spherical coordinates. Specifically, up to a translation moving the point $(\pi/4, \pi/4)$ to the origin, the domain U appearing in the lower right of Figure 2.1 is the square $(0, \pi/2) \times (0, \pi/2)$ in the $u_1 = \phi$ (azumuthal angle) and $u_2 = \theta$ (polar angle) plane. As the translation suggests, the point illustrated corresponds to $(\phi, \theta) = (\pi/4, \pi/4)$. That is, $p = (1/2, 1/2, 1/\sqrt{2})$.

(a) Find the explicit formulas for the (irregularly shaped) domain W in terms of the radius δ.

(b) Generalize your result of part **(a)** to give a formula that works for any reasonably small geodesic disk around a general point p in the first octant of $\mathbb{S}^2$.

Note that in this particular example the "boundary" of $\tilde{X}(B_\delta(\mathbf{0}))$ happens to be a circle in $\mathbb{S}^2$, and in fact the parameterization $\tilde{X}$ can be easily used to find a strict intial parameterization according to Definition 1 of a geodesic[3] disk around the point p. In general, the intersection of a cylinder $\Sigma_{\delta,h}(p, N_p)$ with a local graph will not coincide with the intersection of any ambient ball $B_\delta(p) \subset \mathbb{R}^3$ with the graph/surface in question.

[3]In this discussion, I'm using both terms "boundary," for boundary of a surface, and "geodesic," for paths of locally shortest distance on a surface, in an informal manner. We should discuss the meaning of these words more carefully later.

Prefatory Note: The following was written **before** much of the material above and before sections 2.3 and 2.4 in particular. Definition 5 given below is precisely Definition 1 above. If one has read and understood the material above, then one is perhaps in a pretty good position to read what is below and the main assertion of section 2.4 and the accompanying proof in particular. On the other hand, the material below does not depend directly on sections 2.3 and 2.4. The main assertion of section 2.4 is that the graph of a C^k real valued function defined on an open set $U \subset \mathbb{R}^2$ is a surface according to Definition 1, the "strict" definition of a surface, above. Putting this result together with the pillbox theorem, Theorem 6, above yields that Definition 1 and Definition 2 are equivalent. The reader may also wish to temporarily skip some of the material below and section 2.4 in particualar, as we take a different and in some ways more versatile approach to the same material below. Among other things the alternative approach yields a proof of Theorem 5 on overlapping parameterizations stated above. For those interested in taking the alternative route directly, the discussion starts in section 2.5 below.

2.3 Secondary definitions

For the purposes of this course, I am primarily interested in surfaces embedded in $\mathbb{R}^3$. Here is a definition:

Definition 5. Given $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$ set $\mathcal{S} \subset \mathbb{R}^3$ considered as a topological subspace of $\mathbb{R}^3$ is an **embedded C^k surface** if for each $p \in \mathcal{S}$, there is some $\epsilon > 0$ and a function $X \in C^k(\mathbb{R}^2 \to \mathbb{R}^3)$ for which the following hold:

(S0) $X(0,0) = p$,

(S1) $X : \mathbb{R}^2 \to X(\mathbb{R}^2) \cap B_\epsilon(p)$ is a homeomorphism, and

(S2) For each $\mathbf{u} \in \mathbb{R}^2$ the differential map $dX_{\mathbf{u}} : \mathbb{R}^2 \to \mathbb{R}^3$ has rank 2.

This definition has elements from various definitions in the liturature, but it does not seem to be identical to any definition I have read. Accordingly, one of the first tasks I would like to undertake is examining various equivalent formulations of individual conditions described in the definition and the equivalence (or non-equivalence) of the overall formulation with various alternative formulations. The key to most of these considerations is contained in the following section.

2.4 Graphs and local graphs

If U is an open subset of $\mathbb{R}^2$ and $f \in C^k(U)$, then

$$\mathcal{G} = \{(x_1, x_2, f(x_1, x_2)) \in \mathbb{R}^3 : (x_1, x_2) \in U\}$$

is called the **graph** of the real valued function f. Given $p \in \mathcal{G}$, we can certainly take $\epsilon > 0$ small enough so that $B_{2\epsilon}(\xi(p)) \subset U$ where $\xi(p) = (p_1, p_2)$ and $p = (p_1, p_2, f(p_1, p_2))$. For any such ϵ and a direction $\mathbf{v} = (v_1, v_2) \in \mathbb{S}^1$ there is a well-defined C^1 curve parameterized by

$$\gamma(t) = (\xi(p) + t\mathbf{v}, f(\xi(p) + t\mathbf{v})) = (p_1 + tv_1, p_2 + tv_2, f(\xi(p) + t\mathbf{v})).$$

This curve, say

$$\Gamma = \Gamma(\mathbf{v}) = \{\gamma(t) : -2\epsilon < t < 2\epsilon\}, \tag{2.2}$$

lies in the intersection of $\mathcal{G}$ with the plane

$$P = P(\mathbf{v}) = \{p + \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot (\mathbf{v} \times \mathbf{e}_3) = 0\}$$

passing through p and orthogonal to $(-v_2, v_1, 0)$ where $\mathbf{v}^\perp = (-v_2, v_1)$. This curve as we have it parameterized in (2.2) is precisely the intersection $\Gamma \cap P \cap \mathcal{C}$ where $\mathcal{C} = \{(x_1, x_2, x_3) : (x_1 - p_1)^2 + (x_2 - p_2)^2 < \epsilon^2\}$ is the solid cylinder[4] determined by $\partial B_\epsilon(\xi(p)) \subset Q = \{(x_1, x_2, p_3) : (x_1, x_2) \in \mathbb{R}^2\}$.

Exercise 2.5. Parameterize a potentially larger version of Γ in (2.2) giving the connected component of $\mathcal{G} \cap P$ containing p.

Exercise 2.6. Show that for any $\epsilon > 0$ there is some $\delta > 0$ for which $\gamma(t) \notin \mathcal{G} \cap \partial B_\epsilon(p)$ for $|t| < \delta$.

With Definition 5 in mind, we are now going to consider $\mathcal{G} \cap \partial B_\epsilon(p)$ for certain small values of ϵ and $\mathcal{G} \cap \partial B_\epsilon(p) \cap P(\mathbf{v})$ in particular. Exercise 2.6 suggests a natural separation in the values $t < 0$ for which there might hold $\gamma(t) \in \mathcal{G} \cap \partial B_\epsilon(p) \cap P(\mathbf{v})$ and values $t > 0$ for which $\gamma(t) \in \mathcal{G} \cap \partial B_\epsilon(p) \cap P(\mathbf{v})$. In fact, we are only interested in the values $t > 0$, meaning $\delta < t \le \epsilon$, though we do not emphasize this distinction below or use it in any very important way. If we wanted to, however, we could replace Γ with

$$\Gamma_+ = \Gamma_+(\mathbf{v}) = \{\gamma(t) : \delta < t < 2\epsilon\}. \tag{2.3}$$

Note that $\gamma(0) = p$. By continuity $|\gamma(t) - p| < \epsilon$ for all t small enough, and $|\gamma(t) - p| \ge t \ge \epsilon$ for all t with $\epsilon \le t < 2\epsilon$. That is, $\gamma(t) \in B_\epsilon(p) \cap \mathcal{G}$ for t small and $\gamma(t) \in \mathcal{G} \backslash B_\epsilon(p)$ for $\epsilon \le t < 2\epsilon$. Consequently, the numbers

$$a = \sup\{\alpha \in (0, 2\epsilon) : \gamma(t) \in B_\epsilon(p),\ 0 \le t \le \alpha\} \qquad \text{and} \tag{2.4}$$

$$b = \inf\{\alpha \in (0, 2\epsilon) : \gamma(t) \in \mathcal{G} \backslash B_\epsilon(p),\ \beta \le t < 2\epsilon\} \tag{2.5}$$

are well-defined and satisfy $0 < a \le b \le \epsilon$. By continuity furthermore we know $|\gamma(a) - p| = \epsilon = |\gamma(b) - p|$. That is, $\gamma(a), \gamma(b) \in \partial B_\epsilon(p)$. In particular, $a^2 + [f(\xi(p) + a\mathbf{v}) - f(\xi(p))]^2 = \epsilon^2$ or

$$f(\xi(p) + a\mathbf{v}) = f(\xi(p)) \pm \sqrt{\epsilon^2 - a^2}. \tag{2.6}$$

See Figure 2.2. Let us consider the possibility

[4] There is no fundamental contradiction in considering spheres and cylinders as subsets of $\mathbb{R}^3$ in this discussion of the definition of the term **surface**. We may wish to be a little careful about referring to such sets as "surfaces." Such reference nominally requires justification; see Exercise 2.20 and Exercise 2.21.

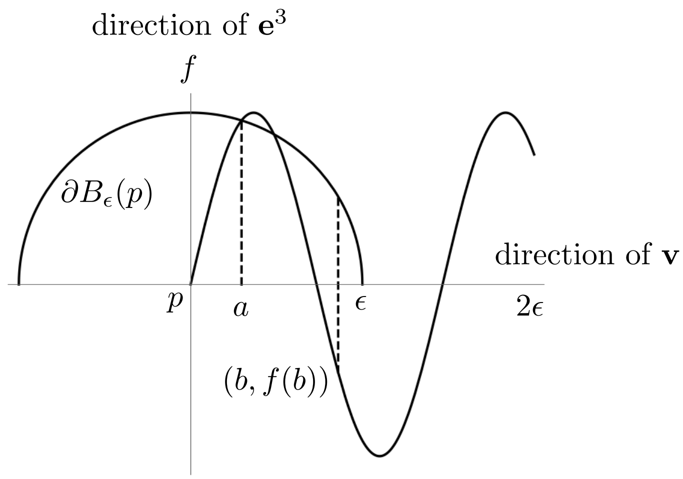

Figure 2.2: Extrinsically radial curve through a point p on a graph in relation to the sphere $\partial B_\epsilon(p)$.

$$f(\xi(p) + a\mathbf{v}) = f(\xi(p)) + \sqrt{\epsilon^2 - a^2}.$$

Then there is some a_0 with $0 < a_0 < a$ for which

$$Df(\xi(p) + a_0\mathbf{v}) \cdot \mathbf{v} = \frac{f(\xi(p) + a\mathbf{v}) - f(\xi(p))}{a} = \frac{\sqrt{\epsilon^2 - a^2}}{a} \geq 0. \tag{2.7}$$

We also have

$$f(\xi(p) + b\mathbf{v}) = f(\xi(p)) \pm \sqrt{\epsilon^2 - b^2},$$

and $f(\xi(p) + b\mathbf{v}) \leq f(\xi(p)) + \sqrt{\epsilon^2 - b^2}$.

Considering further the possibility $a < b$, we see first of all $a < \epsilon$ and the inequality in (2.7) is strict. Beyond this, we can say there is some b_0 with $a < b_0 < b$ for which

$$\begin{aligned} Df(\xi(p) + b_0\mathbf{v}) \cdot \mathbf{v} &= \frac{f(\xi(p) + b\mathbf{v}) - f(\xi(p) + a\mathbf{v})}{b - a} \\ &\leq \frac{\sqrt{\epsilon^2 - b^2} - \sqrt{\epsilon^2 - a^2}}{b - a} \\ &< 0. \end{aligned}$$

Under these assumptions then we have values a_0 and b_0 for which

$$\begin{aligned}
|[Df(\xi(p) + a_0\mathbf{v}) - Df(\xi(p) + b_0\mathbf{v})] \cdot \mathbf{v}|& \\
&= [Df(\xi(p) + a_0\mathbf{v}) - Df(\xi(p) + b_0\mathbf{v})] \cdot \mathbf{v} \\
&\geq \frac{\sqrt{\epsilon^2 - a^2}}{a} + \frac{\sqrt{\epsilon^2 - a^2} - \sqrt{\epsilon^2 - b^2}}{b - a}. \qquad (2.8)
\end{aligned}$$

The expression (2.8) decreases as b tends downward to a with limiting value

$$\frac{\sqrt{\epsilon^2 - a^2}}{a} + \frac{a}{\sqrt{\epsilon^2 - a^2}}.$$

Exercise 2.7. Explain why this is the limiting value. (Of course you can simply give an analytic explanation, but you can also use the illustration in Figure 2.2.)

It follows that

$$|[Df(\xi(p) + a_0\mathbf{v}) - Df(\xi(p) + b_0\mathbf{v})] \cdot \mathbf{v}| \geq \frac{\sqrt{\epsilon^2 - a^2}}{a} + \frac{a}{\sqrt{\epsilon^2 - a^2}}$$

independent of b subject only to $0 < a < b \leq \epsilon$. The bound on the right is of the form $A + 1/A$ where $A = \sqrt{\epsilon^2 - a^2}/a$. From this we see that for $0 < a < \epsilon$ the quantity A monotonically decreases taking (all) values on $(0, \infty)$. The function $A + 1/A$ on the other hand takes a minimum value of 2 for $0 < A < \infty$ independent of ϵ. From this we conclude there are values a_0 and b_0 with $0 < a_0 < b_0 < \epsilon$ satisfying

$$|[Df(\xi(p) + a_0\mathbf{v}) - Df(\xi(p) + b_0\mathbf{v})] \cdot \mathbf{v}| \geq 2. \qquad (2.9)$$

This is a problem when ϵ is small because in that case both gradients $Df(\xi(p) + a_0\mathbf{v})$ and $Df(\xi(p) + b_0\mathbf{v})$ are close to $Df(\xi(p))$. More explicitly, there is some ϵ so that $|Df(\mathbf{u}) - Df(\xi(p))| < 1$ whenever $|\mathbf{u} - \xi(p)| < \epsilon$. And for such a value of ϵ

$$\begin{aligned}
|[Df(\xi(p) + a_0\mathbf{v}) - Df(\xi(p) + b_0\mathbf{v})] \cdot \mathbf{v}|& \\
&\leq |Df(\xi(p) + a_0\mathbf{v}) - Df(\xi(p))| + |Df(\xi(p)) - Df(\xi(p) + b_0\mathbf{v})| \\
&< 2,
\end{aligned}$$

and this contradicts (2.9). Our initial conclusion from this is $a = b$.

In summary, the positive numbers a and b given in (2.4) and (2.5) are always well-defined and always satisfy $0 < a \le b \le \epsilon$. We have made a choice of sign in (2.6) and assumed further $a < b$ to get a contradiction. The contradiction means $a = b$.

Without the assumption $a < b$, we do not get the contradiction, but the alternative $a = b$ is precisely the same conclusion. Note this observation applies in the particular case in which $a = \epsilon$ and the inequality in (2.6) becomes equality. That is to say, the overall argument leading to a contradiction is not valid, but $a = \epsilon$ implies $a = b = \epsilon$, so the same conclusion holds in a very specific and strong sense. It will be convenient below to distinguish this possibility as a separate case, $\Gamma \cap \partial B_\epsilon(p) \ni (\xi(p) + \epsilon\mathbf{v}, p_3)$ or more precisely $\Gamma_+ \cap \partial B_\epsilon(p) = \{(\xi(p) + \epsilon\mathbf{v}, p_3)\}$, that is $b = \epsilon$ in which case we have shown also $a = b = \epsilon$.

Were we to consider the other sign in (2.6) that is,

$$f(\xi(p) + a\mathbf{v}) = f(\xi(p)) - \sqrt{\epsilon^2 - a^2}, \tag{2.10}$$

and we ruled out the trivial case $a = \epsilon$, and assume further by contradiction that $a < b$, then we can arrive at a (similar) contradiction. If you want to make sure you understand the reasoning above, carrying out the details of such an argument might be a way to do that.

Exercise 2.8. Give the details leading to a contradiction under the assumption $a < b$ and $f(\xi(p) + a\mathbf{v}) = f(\xi(p)) - \sqrt{\epsilon^2 - a^2} < f(\xi(p))$.

Exercise 2.9. Give a simpler argument leading to the conclusion that for $\epsilon > 0$ small enough and each $\mathbf{v} \in \mathbb{S}^1$ there holds

$$\Gamma_+(\mathbf{v}) \cap \partial B_\epsilon(p) = \{\gamma(a)\},$$

that is, $\Gamma_+(\mathbf{v}) \cap \partial B_\epsilon(p)$ consists of a single point $\gamma(a)$ with three distinct cases possible:

(i) $0 < a < \epsilon$ and $\gamma(a) = (\xi(p) + a\mathbf{v}, p_3 - \sqrt{\epsilon^2 - a^2})$,

(ii) $a = \epsilon$ and $\gamma(a) = (\xi(p) + \epsilon\mathbf{v}, p_3)$, or

(iii) $0 < a < \epsilon$ and $\gamma(a) = (\xi(p) + a\mathbf{v}, p_3 + \sqrt{\epsilon^2 - a^2})$.

We have shown that for $\epsilon > 0$ small enough, there is a well-defined function $a = a(\mathbf{v}) > 0$ such that $\mathcal{G} \cap \partial B_\epsilon(\mathbf{p})$ is given precisely by the set

$$\mathcal{G} \cap \partial B_\epsilon(\mathbf{p}) = \{(\xi(p)p + a(\mathbf{v})\mathbf{v},\ f(\xi(p) + a(\mathbf{v})\mathbf{v})) \in \mathbb{R}^3 : \mathbf{v} \in \mathbb{S}^1\}.$$

One might well hope $a : \mathbb{S}^1 \to \mathbb{R}$ is a function with some regularity so that, for example, $\mathcal{G} \cap \partial B_\epsilon(\mathbf{p})$ is a C^k simple closed curve. We proceed to verify that this is the case.

Some regularity may be verified directly without the use of the inverse or implicit function theorems.

Exercise 2.10. Verify directly (without using the inverse function theorem) that the function $a : \mathbb{S}^1 \to \mathbb{R}$ defined above is continuous, so that

$$B = \{\xi(p) + a(\mathbf{v})\mathbf{v} : \mathbf{v} \in \mathbb{S}^1\}$$

is a continuous simple closed (star shaped) planar curve.

Consider first the situation in which $a = a(\mathbf{v}) < \epsilon$ and

$$f(\xi(p) + a\mathbf{v}) = f(\xi(p)) - \sqrt{\epsilon^2 - a^2} < f(p).$$

This is described as case **(i)** above. Note first there is some a_0 with $0 < a_0 < a$

$$Df(\xi(p) + a_0\mathbf{v}) \cdot \mathbf{v} = \frac{f(\xi(p) + av) - f(\xi(p))}{a} = -\frac{\sqrt{\epsilon^2 - a^2}}{a} < 0.$$

We claim

$$Df(\xi(p) + a\mathbf{v}) \cdot \mathbf{v} < Dg(\xi(p) + a\mathbf{v}) \cdot \mathbf{v} = \frac{a}{\sqrt{\epsilon^2 - a^2}} \tag{2.11}$$

where $g : B_\epsilon(\xi(p)) \to \mathbb{R}$ by

$$g(x_1, x_2) = p_3 - \sqrt{\epsilon^2 - |(x_1, x_2) - \xi(p)|^2}$$

is the real valued function whose graph is the lower hemisphere determined by $B_\epsilon(p)$. To see this, note first that

$$Dg(x_1, x_2) = \frac{(x_1 - p_1, x_2 - p_2)}{\sqrt{\epsilon^2 - |(x_1, x_2) - \xi(p)|^2}}.$$

Therefore,

$$Dg(\xi(p)+a\mathbf{v})\cdot\mathbf{v}=\frac{a\mathbf{v}}{\sqrt{\epsilon^2-|a\mathbf{v}|^2}}\cdot\mathbf{v}=\frac{a}{\sqrt{\epsilon^2-a^2}}$$

as asserted. On the other hand, the inequality $f(\xi(p)+t\mathbf{v})\geq g(\xi(p)+t\mathbf{v})$ for $0<t\leq a$ implies

$$Df(\xi(p)+a\mathbf{v})\cdot\mathbf{v}\leq Dg(\xi(p)+a\mathbf{v})\cdot\mathbf{v}. \tag{2.12}$$

If equality holds in (2.12), then

$$Df(\xi(p)+a\mathbf{v})\cdot\mathbf{v}=\frac{a}{\sqrt{\epsilon^2-a^2}}>0,$$

and

$$\begin{aligned}|Df(\xi(p)+a_0\mathbf{v})\cdot\mathbf{v}-Df(\xi(p)+a\mathbf{v})\cdot\mathbf{v}| &= \left|-\frac{\sqrt{\epsilon^2-a^2}}{a}-\frac{a}{\sqrt{\epsilon^2-a^2}}\right| \\ &= \frac{\sqrt{\epsilon^2-a^2}}{a}+\frac{a}{\sqrt{\epsilon^2-a^2}} \\ &\geq 2\end{aligned}$$

as shown above. Again, this is a contradiction since we have taken ϵ small enough so that $|Df(\xi(p)+a_0\mathbf{v})-Df(\xi(p))|$ and $|Df(\xi(p))-Df(\xi(p)+a\mathbf{v})|$ are both smaller than 1. Thus, we have established the strict inequality (2.11). This means in particular

$$Dg(\xi(p)+a\mathbf{v})\cdot\mathbf{v}-Df(\xi(p)+a\mathbf{v})\cdot\mathbf{v}=\frac{a}{\sqrt{\epsilon^2-a^2}}-Df(\xi(p)+a\mathbf{v})\cdot\mathbf{v}>0. \tag{2.13}$$

Now let's fix a particular $V\in\mathbb{S}^1$ with $a(V)<\epsilon$ as suggested above in case **(i)**, find a number Θ for which $V=(\cos\Theta,\sin\Theta)$, and consider a mapping $\psi:(0,\epsilon)\times\mathbb{R}\to\mathbb{R}^2$ by

$$\psi(r,\theta)=(g(\xi(p)+r\mathbf{v})-f(\xi(p)+r\mathbf{v}),\theta)$$

where $\mathbf{v}=(\cos\theta,\sin\theta)$. We need to properly translate the assertion of (2.13) into this current broader context. That translation is

$$\begin{aligned}&Dg(\xi(p)+a(V)V)\cdot V-Df(\xi(p)+a(V)V)\cdot V \\ &\qquad=\frac{a}{\sqrt{\epsilon^2-a^2}}-Df(\xi(p)+a(V)V)\cdot V>0.\end{aligned} \tag{2.14}$$

Notice $(a(V), \Theta)$ is in the open strip domain $\Sigma = (0, \epsilon) \times \mathbb{R}$ with $\psi(a(V), \Theta) = (0, \Theta)$, and $\psi = (\psi_1, \psi_2) \in C^k(\Sigma \to \mathbb{R}^2)$. Furthermore,

$$D\psi = \begin{pmatrix} Dg(\xi(p) + r\mathbf{v}) \cdot \mathbf{v} - Df(\xi(p) + r\mathbf{v}) \cdot \mathbf{v} & \partial\psi_1/\partial\theta \\ 0 & 1 \end{pmatrix}$$

so

$$\det D\psi(a(V), \Theta) = \frac{a}{\sqrt{\epsilon^2 - a^2}} - Df(\xi(p) + a(V)V) \cdot V > 0.$$

This means ψ has a local C^k inverse function $\psi\big|_{U_0}^{-1} : \psi(U_0) \to U_0$ where U_0 is some open subset of Σ with $(a(V), \Theta) \in U_0$. We write

$$\psi\big|_{U_0}^{-1} = (w_1, w_2)$$

denoting the component functions of the local inverse by $w_j : \psi(U_0) \to \mathbb{R}$ for $j = 1, 2$.

Focusing on the image point

$$(0, \Theta) = \psi(a(V), \Theta)$$

in the open set $\psi(U_0)$, we take some $\delta > 0$ with $B_\delta(0, \Theta) \subset \psi(U_0)$ and define a function $w \in C^k(\Theta - \delta, \Theta + \delta)$ by

$$w(\theta) = w_1(0, \theta). \tag{2.15}$$

See Figure 2.3. Let us note at this point that $U_0 \subset \Sigma \subset \{(r, \theta) : r > 0\}$. From this we see immediately that

$$w(\theta) > 0 \qquad \text{for} \qquad \Theta - \delta < \theta < \Theta + \delta. \tag{2.16}$$

That is, w is a strictly positive function.

On the one hand, because $\psi\big|_{U_0}^{-1}$ is an inverse for $\psi\big|_{U_0}$ we have

$$\begin{aligned}(r, \theta) = \Big(&w_1(g(\xi(p) + r\mathbf{v}) - f(\xi(p) + r\mathbf{v}), \theta), \\ &w_2(g(\xi(p) + r\mathbf{v}) - f(\xi(p) + r\mathbf{v}), \theta)\Big)\end{aligned} \tag{2.17}$$

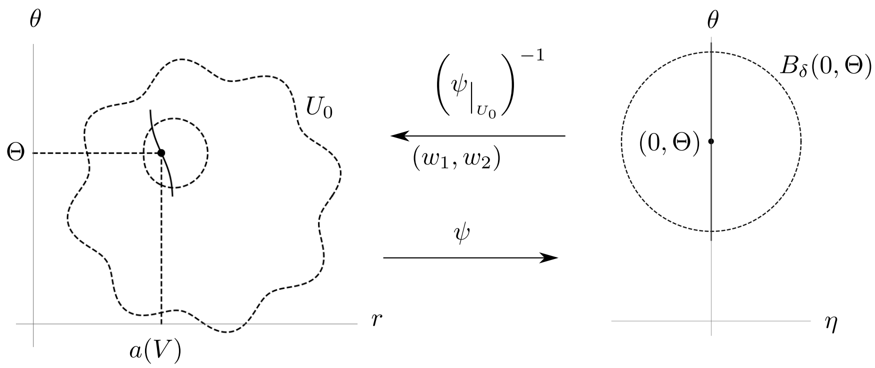

Figure 2.3: Mapping from which a local radius function $r = w(\theta)$ may be determined implicitly using the inverse function theorem.

for $(r, \theta) \in U_0$ and $\mathbf{v} = (\cos\theta, \sin\theta)$ as usual. Applying this identity at the particular point $(r, \theta) = (a(V), \Theta)$ where $V = (\cos\Theta, \sin\Theta)$,

$$(w_1(0, \Theta), w_2(0, \Theta)) = (a(V), \Theta). \tag{2.18}$$

This tells us in particular that $w(\Theta) = w_1(0, \Theta) = a(V)$.

The reverse composition of $\psi^{-1}\big|_{U_0}$ and ψ applied to a point $(\eta, \theta) \in B_\delta(0, \Theta)$ yields the identity

$$\begin{aligned}(\eta, \theta) &= \psi(w_1(\eta, \theta), w_2(\eta, \theta)) \\ &= \Big(g(\xi(p) + w_1(\eta, \theta)(\cos w_2(\eta, \theta), \sin w_2(\eta, \theta)) \\ &\qquad - f(\xi(p) + w_1(\eta, \theta)(\cos w_2(\eta, \theta), \sin w_2(\eta, \theta)), \\ &\qquad\qquad w_2(\eta, \theta) \Big)\end{aligned} \tag{2.19}$$

We may observe first from this the obvious identity $w_2(\eta, \theta) \equiv \theta$ independent of η. Note this observation was already present in (2.17) in a seemingly special case.

Substituting $w_2(\eta, \theta) = \theta$ in the first component of (2.19) we obtain

$$g(\xi(p) + w_1(\eta, \theta)(\cos\theta, \sin\theta)) - f(\xi(p) + w_1(\eta, \theta)(\cos\theta, \sin\theta)) = \eta.$$

Specializing to $\eta = 0$, the resulting relation will hold at least for $\Theta - \delta < \theta < \Theta + \delta$, namely

$$g(\xi(p) + w_1(0,\theta)(\cos\theta, \sin\theta)) - f(\xi(p) + w_1(0,\theta)(\cos\theta, \sin\theta)) = 0. \quad (2.20)$$

Recalling that $w(\theta) = w_1(0,\theta)$ (2.20) can be written as

$$g(\xi(p) + w(\theta)(\cos\theta, \sin\theta)) - f(\xi(p) + w(\theta)(\cos\theta, \sin\theta)) = 0.$$

Since we have arranged to have $w(\theta) > 0$ as indicated in (2.16) and we have also shown there is for each $\mathbf{v} \in \mathbb{S}^1$ exactly one positive value $a = a(\mathbf{v})$ for which

$$g(\xi(p) + a\mathbf{v}) - f(\xi(p) + a\mathbf{v}) = 0,$$

it must be the case that $w = w(\theta) = a = a(\mathbf{v})$ when $\mathbf{v} = (\cos\theta, \sin\theta)$ as is the case here.

This means the image

$$\{\xi(p) + a(\mathbf{v}) : \mathbf{v} = (\cos\theta, \sin\theta),\ \Theta - \delta < \theta < \Theta + \delta\} \qquad (2.21)$$

is a C^k curve parameterized by

$$\beta(\theta) = \xi(p) + w(\theta)(\cos\theta, \sin\theta).$$

We pause here to recall the assumption of case **(i)** that $a(V) < \epsilon$ and

$$f(\xi(p) + a(V)V) = f(\xi(p)) - \sqrt{\epsilon^2 - a(V)^2}.$$

Were we to obtain a similar conclusion concerning the image (2.21) in cases **(ii)** and **(iii)**, it would follow that the projection

$$\{\xi(p) + a(\mathbf{v})\mathbf{v} : \mathbf{v} \in \mathbb{S}^1\}$$

is a simple closed planar curve bounding a star shaped domain $W \subset \mathbb{R}^2$, and the domain W is star shaped with respect to the point $\xi(p) = (p_1, p_2) \in W$. Also,

$$\mathcal{G} \cap \partial B_\epsilon(p) = \{(\xi(p) + a(\mathbf{v})\mathbf{v}, f(\xi(p) + a(\mathbf{v})\mathbf{v})) : \mathbf{v} \in \mathbb{S}^1\}$$

is a simple closed space curve on $\partial B_\epsilon(p)$.

You may note that the argument given above for case **(i)** gets derailed pretty quickly in case **(ii)** where $a(V) = \epsilon$. This is because $\partial B_\epsilon(p)$ is not given locally as the graph of a function of x_1 and x_2 at the point $(\xi(p) +$

$a(V)V, f(\xi(p) + a(V)V) = (\xi(p) + a(V)V, p_3)$. For this reason we take a somewhat similar but in a way rather different approach.

We again fix $V \in \mathbb{S}^1$ and take $\Theta \in \mathbb{R}$ with $V = (\cos\Theta, \sin\Theta)$. We also define a mapping ψ, but the domain and definition are a bit different. Here we take $\psi : (0, 2\epsilon) \times \mathbb{R} \to \mathbb{R}^2$ by

$$\begin{aligned}\psi(r,\theta) &= (|p - (\xi(p) + r\mathbf{v}, f(\xi(p) + r\mathbf{v}))|, \theta)\\ &= \left(\sqrt{r^2 + (p_3 - f(\xi(p) + r\mathbf{v}))^2}, \theta\right)\end{aligned}$$

where as usual $\mathbf{v} = \mathbf{v}(\theta) = (\cos\theta, \sin\theta)$. The point of interest now is (ϵ, Θ). When we evaluate ψ at (ϵ, Θ) we get (ϵ, Θ). If this causes some confusion, perhaps it is worthwhile to replace ψ with ψ_0 given by something like

$$\psi_0(r,\theta) = \left(\epsilon - \sqrt{r^2 + (p_3 - f(\xi(p) + r\mathbf{v}))^2}, \theta\right).$$

Then $\psi_0(\epsilon, \Theta) = (0, \Theta)$. But let's see how we do with ψ. We compute

$$D\psi = \begin{pmatrix} \dfrac{r - (p_3 - f(\xi(p) + r\mathbf{v}))Df(\xi(p) + r\mathbf{v}) \cdot \mathbf{v}}{\sqrt{r^2 + (p_3 - f(\xi(p) + r\mathbf{v}))^2}} & \dfrac{\partial\psi_1}{\partial\theta} \\ 0 & 1 \end{pmatrix}.$$

There's clearly some similarity between the new ψ for case **(ii)** and the function ψ used above for case **(i)** evident here. Evaluating at $(r, \theta) = (\epsilon, \Theta)$ we find

$$\det D\psi(\epsilon, \Theta) = 1 > 0. \tag{2.22}$$

The fact that this value is exactly 1 is a kind of interesting geometric computation which might seem nonintuitive at first; see Appendix A.

The computation does seem to be correct, however, and accordingly there is an open set U_0 with $(\epsilon, \Theta) \in U_0 \subset \{(r, \theta) : r > 0\}$ on which ψ is a homeomorphism onto the image $\psi(U_0)$. Taking $B_\delta(\epsilon, \Theta) \subset \psi(U_0)$ and setting $w(\theta) = w_1(\epsilon, \theta)$ where $\psi\big|_{U_0} = (w_1, w_2)$ and $w_2(\eta, \theta) \equiv \theta$ we obtain a radius function $w \in C^k(\Theta - \delta, \Theta + \delta)$ for which

$$\begin{aligned}&\{\xi(p) + a(\mathbf{v})\mathbf{v} : \mathbf{v} = (\cos\theta, \sin\theta),\ \Theta - \delta < \theta < \Theta + \delta\}\\ &\qquad = \{\xi(p) + w(\theta)(\cos\theta, \sin\theta) : \Theta - \delta < \theta < \Theta + \delta\}.\end{aligned}$$

Along with the treatment of case **(iii)** this establishes that $\mathcal{G} \cap \partial B_r(p)$ is a C^k simple closed curve which projects to a star shaped C^k simple closed curve bounding a domain containing $\xi(p) = (p_1, p_2)$ as described above.

Exercise 2.11. Draw an illustration of the mapping ψ described above for case **(ii)** analogous to the illustration in Figure 2.3.

Exercise 2.12. Give a proof that for $\epsilon > 0$ small $\mathcal{G} \cap \partial B_\epsilon(p)$ is a C^k curve near a point $(\xi(p) + a(V)V, f(\xi(p) + a(V)V))$ in case **(iii)**. (Try to give a simpler proof than the one given above for case **(i)**.)

Finally, consider the domain

$$\{\xi(p) + r(\cos\theta, \sin\theta) : 0 \le r < w(\theta),\ \theta \in \mathbb{R}\} \tag{2.23}$$

on which $X(x_1, x_2) = (x_1, x_2, f(x_1, x_2))$ parameterizes $\mathcal{G} \cap B_\epsilon(p)$. We claim this domain is C^k diffeomorphic to $\mathbb{R}^2$.

Exercise 2.13. Explain why the definition (2.23) is valid even though the function w has values defined in very different ways in cases **(i)**, **(ii)**, and **(iii)** and nominally only in some small intervals $(\Theta - \delta, \Theta + \delta)$ for certain specific values of Θ.

The polar coordinates map $\Psi : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ given by

$$\Psi(r, \theta) = (r\cos\theta, r\sin\theta)$$

has a singularity along the line $r = 0$ but otherwise is a local C^∞ diffeomorphism periodic in θ. One consequence is that any 2π periodic function $f \in C^k(\mathbb{R})$ determines a function $g \in C^k(\overline{A})$ of two variables on any annular region

$$A = \{\xi(p) + r(\cos\theta, \sin\theta) : r > \epsilon,\ \theta \in \mathbb{R}\}$$

where $\epsilon > 0$ and

$$g(\mathbf{x}) = f(\theta)$$

for any θ with $\mathbf{x} = (x_1, x_2) = \xi(p) + |\mathbf{x} - \xi(p)|(\cos\theta, \sin\theta)$. This construction is discussed further in Appendix B where the values of the function g are given the useful form $g(\mathbf{x}) = f \circ \arg(\mathbf{x})$ in terms of a (principal) argument function $\arg : \mathbb{R}^2 \backslash \{0, 0\} \to \mathbb{R}$.

In particular the functions with values

$$\cos(\arg(\mathbf{x})) = \frac{x_1}{|\mathbf{x}|} \qquad \text{and} \qquad \sin(\arg(\mathbf{x})) = \frac{x_2}{|\mathbf{x}|}$$

are well-defined and smooth on $\mathbb{R}^2 \backslash \{(0,0)\}$. The regularity and well-defined property are clear from the formulas of course, but it is useful to keep in mind the relation with the local inverse(s) of the polar coordinates map Ψ. A similar observation applies to $w(\mathbf{e}_2 \cdot \Psi^{-1}(\mathbf{x}))$ for any 2π periodic function w. See Appendix B for further details.

Lemma 2.1. If $W \subset \mathbb{R}^2$ is a star shaped with respect to $\mathbf{0} = (0,0) \in W$ and is C^k in the sense that ∂W is a C^k radial graph, i.e., there exists a positive 2π periodic function $w \in C^k(\mathbb{R})$ such that

$$W = \{r(\cos\theta, \sin\theta) : 0 \le r < w(\theta),\ \theta \in \mathbb{R}\}$$

and

$$\partial W = \{w(\theta)(\cos\theta, \sin\theta) : \theta \in \mathbb{R}\},$$

then W is diffeomorphic to $\mathbb{R}^2$.

Proof: Recall that for any radius $r > 0$ the set $B_r(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < r\}$ is C^∞ diffeomorphic to $\mathbb{R}^2$ by

$$\beta(\mathbf{x}) = \beta(\mathbf{x}; r) = \frac{\mathbf{x}}{r^2 - |\mathbf{x}|^2}.$$

Note the mapping $g \in C^\infty(0, r)$ with

$$g(x) = g(x; r) = \frac{x}{r^2 - x^2}$$

may be associated with the mapping β, and $\beta(\mathbf{x}) = g(|\mathbf{x}|)$. Also,

$$g'(x) = \frac{r^2 + x^2}{r^2 - x^2} > 0.$$

Set

$$m = \min_{\mathbf{v} \in \mathbb{S}^1} a(\mathbf{v}) = \min_{\theta \in \mathbb{R}} w(\theta) \qquad \text{and} \qquad M = \max_{\mathbf{v} \in \mathbb{S}^1} a(\mathbf{v}) = \max_{\theta \in \mathbb{R}} w(\theta).$$

Restricting $\beta = \beta(\mathbf{x}; M)$ to $B_t(\mathbf{0})$ for some fixed t with $0 < t < M$, we obtain a C^∞ diffeomorphism of $B_t(\mathbf{0})$ onto $B_{g(t)}(\mathbf{0})$ with finite limits

$$\lim_{|\mathbf{x}| \to t} g(|\mathbf{x}|; M) = \frac{t}{M^2 - t^2} \qquad \text{and} \qquad \lim_{|\mathbf{x}| \to t} g'(|\mathbf{x}|; M) = \frac{M^2 + t^2}{M^2 - t^2} > 0$$

in particular. Let us focus for a moment on the value $t = m/4$. Then we have a diffeomorphsim of $B_{m/4}(\mathbf{0})$ onto $B_{g(m/4)}(\mathbf{0})$ with associated finite values

$$g(m/4) = \frac{m/4}{M^2 - m^2/16} \qquad \text{and} \qquad g'(m/4) = \frac{M^2 + m^2/16}{M^2 - m^2/16} > 0.$$

Now for $\theta \in \mathbb{R}$ fixed consider the function $h : (0, w(\theta)) \to \mathbb{R}$ by

$$h(x) = h(x; w(\theta)) = \frac{x}{w^2 - x^2}.$$

Since $w = w(\theta) \le M$, we find

$$\begin{aligned} h(m/4) &= \frac{m/4}{w^2 - m^2/16} \ge g(m/4) \qquad \text{and} \\ h'(m/4) &= \frac{w^2 + m^2/16}{w^2 - m^2/16} \ge g'(m/4). \end{aligned} \tag{2.24}$$

Notice that in the last inequality we have used the fact that $g' = g'(x; r)$ is decreasing in r:

$$\frac{\partial}{\partial r} g'(x; r) = -\frac{4rx^2}{(r^2 - x^2)^2} < 0.$$

In fact, by using $g(x; M')$ with some $M' > M$ we could ensure strict inequalities in (2.24). As it stands, those inequalities are also strict for any values of θ for which $w(\theta) < M$.

More importantly, since h' and h'' are both positive with

$$g''(x; r) = \frac{4r^2 x}{(r^2 - x^2)^2},$$

we have $h(x) > g(m/4)$ and $h'(x) > g'(m/4)$ for any $x > m/4$, and these inequalities hold for $x = 3m/4$ in particular which is where we will nominally now use them.

Consider a particular ray $\{x(\cos\theta, \sin\theta) : x > 0\}$. Figure 2.4 illustrates how the graphs of the monotone functions h and g might appear on the overlapping intervals $0 < x < 3m/4$ and $m/4 < x < M$. Each of the functions g and h is smooth (C^∞) on its domain though the dependence on θ arising through the endpoint $w = w(\theta)$ is only C^k; we will consider the consequences of this later. For the moment, considering θ fixed, we seek a smooth function transitioning from the values of g over the interval $0 \le x \le m/4$ to the values of h over the interval $3m/4 \le x < w$ with the transition taking place over the interval $m/4 < x < 3m/4$ with monotonicity and regularity preserved. In order to accomplish this transition, we construct

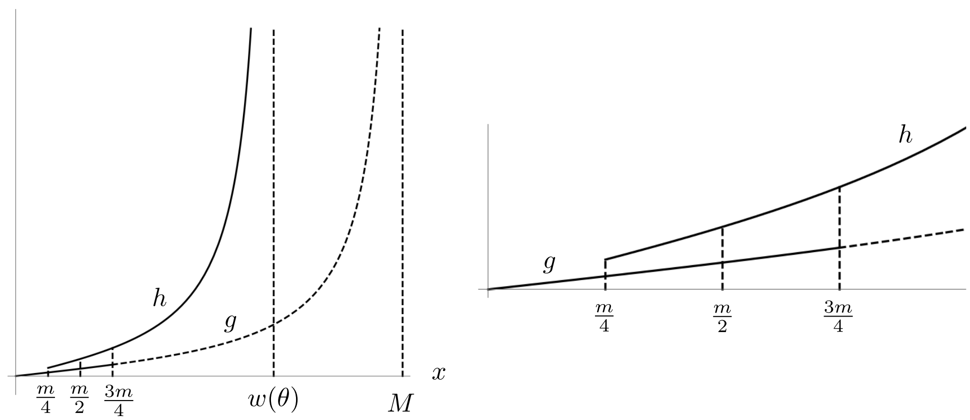

Figure 2.4: Monotone functions giving radial diffeomorphisms from a disk to $\mathbb{R}^2$.

a function $\lambda \in C^\infty(\mathbb{R})$ satisfying

$$\begin{cases} \lambda(x) \equiv 0, & x \leq m/4, \\ \lambda(x) > 0, & m/4 < x < 3m/4, \text{ and} \\ \lambda(x) \equiv 1, & x \geq 3m/4. \end{cases} \tag{2.25}$$

We can obtain such a function as a mollification of the characteristic function with values

$$\chi_{(m/2,\infty)}(x) = \begin{cases} 0, & x \leq m/2 \\ 1, & x > m/2. \end{cases}$$

Specifically we can take

$$\lambda(x) = \int_{\xi \in \mathbb{R}} \chi_{(m/2,\infty)}(\xi)\mu(x - \xi) \tag{2.26}$$

where $\mu = \mu_{m/4}$ is the mollifier μ_δ with $\delta = m/4$ given by

$$\mu(x) = \frac{1}{\delta I_0}\, \mu_0\left(\frac{x}{\delta}\right); \qquad \mu_0(x) = \begin{cases} 0, & |x| \geq \delta \\ e^{1/(1-x^2)}, & x < \delta \end{cases}; \qquad I_0 = \int_{\mathbb{R}} \mu_0.$$

Exercise 2.14. Verify the function λ as defined by (2.26) satisfies $\lambda \in C^\infty(\mathbb{R})$ and has the properties (2.25).

Now we can "glue" the functions h and g together or "transition" from g to h by setting

$$G(x) = (1 - \lambda(x))g(x) + \lambda(x)h(x).$$

In this way we obtain $G \in C^\infty(0, w(\theta))$ with

$$\begin{cases} G(x) \equiv g(x), & 0 \le x \le m/4 \\ G(x) \equiv h(x), & x > 3m/4 \end{cases}$$

and

$$G'(x) = (1 - \lambda)g' + \lambda h' + \lambda'(g + h) > 0.$$

Finally we consider the function

$$\psi : \{x(\cos\theta, \sin\theta) : 0 \le x < w(\theta),\ \theta \in \mathbb{R}\} \to \mathbb{R}^2$$

with values

$$\psi(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} = (0,0) \\ \\ G(|\mathbf{x}|)\dfrac{\mathbf{x}}{|\mathbf{x}|}, & \mathbf{x} \ne (0,0). \end{cases}$$

Exercise 2.15. Show $\psi : W \to \mathbb{R}^2$ is one-to-one and onto.

Note that for $|\mathbf{x}| < m/4$ we have $G(|\mathbf{x}|) \equiv g(|\mathbf{x}|)$ and

$$\psi(\mathbf{x}) = \beta(\mathbf{x}) = \beta(\mathbf{x}; r)$$

with $r = M$. Thus, ψ is a local C^∞ diffeomorphism at $\mathbf{x} = 0$. In fact, for $|\mathbf{x}| < m/4$ we have

$$\psi(\mathbf{x}) = \frac{\mathbf{x}}{M^2 - |\mathbf{x}|^2}$$

and

$$D\psi = \begin{pmatrix} \dfrac{1}{M^2 - |\mathbf{x}|^2} + 2\dfrac{x_1^2}{(M^2 - |\mathbf{x}|^2)^2} & 2\dfrac{x_1 x_2}{(M^2 - |\mathbf{x}|^2)^2} \\ \\ 2\dfrac{x_1 x_2}{(M^2 - |\mathbf{x}|^2)^2} & \dfrac{1}{M^2 - |\mathbf{x}|^2} + 2\dfrac{x_2^2}{(M^2 - |\mathbf{x}|^2)^2} \end{pmatrix}.$$

In particular, for $\mathbf{x} = 0$ we have $D\psi(0,0)$ is a multiple of the identity matrix, and in general for $0 < |\mathbf{x}| \le m/4$

$$\det D\psi = \frac{1}{(M^2 - |\mathbf{x}|^2)^2} + 2\frac{|\mathbf{x}|^2}{(M^2 - |\mathbf{x}|^2)^3} > 0.$$

We claim $\det D\psi(\mathbf{x}) > 0$ in general for $0 \le |\mathbf{x}| < w \circ \arg(\mathbf{x})$, that is, for $\mathbf{x} \in W$. If this assertion can be verified, then for each $q \in \mathbb{R}^2$, there is some $\delta > 0$ for which

$$\psi\Big|_{B_\delta(\psi^{-1}(q))} : B_\delta(\psi^{-1}(q)) \to \psi\left(B_\delta(\psi^{-1}(q))\right)$$

is a C^k diffeomorphism and $\psi^{-1} \in C^k(\mathbb{R}^2)$.

We first show $\det D\psi(x_1, 0) > 0$ when $0 < x_1 < w(0) = w \circ \arg(x_1, 0)$. Let us begin with a review of the somewhat complicated construction of ψ: We have two functions g and h both defined on $[0, w) = [0, w \circ \arg(\mathbf{x})) \subset [0, M)$ with $g(0) = h(0) = 0$ and

$$g(x) = \frac{x}{M^2 - x^2} \qquad \text{and} \qquad h(x) = \frac{x}{w^2 - x^2}.$$

It is important now that $w = w(\theta) = w \circ \arg(\mathbf{x})$ contains an additional dependence, though we may still temporarily think of x here and $\mathbf{x}$ as indepedndent of one another. It may be remarked that $w \circ \arg(\mathbf{x})$ is a C^k radial function with domain $W \backslash \{(0, 0)\}$ as discussed in Appendix B.

We have also a C^∞ function λ with $\lambda(x) \equiv 0$ for $x < m/4$, $\lambda(x) \equiv 1$ for $x \ge 3m/4$, and $\lambda'(x) > 0$ for $m/4 < x < 3m/4$. Our mapping ψ then takes the form

$$\begin{aligned} \psi(\mathbf{x}) &= \left((1 - \lambda(|\mathbf{x}|))\frac{|\mathbf{x}|}{M^2 - |\mathbf{x}|^2} + \lambda(|\mathbf{x}|)\frac{|\mathbf{x}|}{w^2 - |\mathbf{x}|^2}\right)\frac{\mathbf{x}}{|\mathbf{x}|} \\ &= \left(\frac{1 - \lambda(|\mathbf{x}|)}{M^2 - |\mathbf{x}|^2} + \frac{\lambda(|\mathbf{x}|)}{w^2 - |\mathbf{x}|^2}\right)\mathbf{x}. \end{aligned} \tag{2.27}$$

We introduce the coordinate functions with

$$\psi_1(\mathbf{x}) = \left(\frac{1 - \lambda(|\mathbf{x}|)}{M^2 - |\mathbf{x}|^2} + \frac{\lambda(|\mathbf{x}|)}{w^2 - |\mathbf{x}|^2}\right)x_1$$

and

$$\psi_2(\mathbf{x}) = \left(\frac{1 - \lambda(|\mathbf{x}|)}{M^2 - |\mathbf{x}|^2} + \frac{\lambda(|\mathbf{x}|)}{w^2 - |\mathbf{x}|^2}\right)x_2$$

and express them as $\psi_j(\mathbf{x}) = H_j(|\mathbf{x}|, \mathbf{x})$ for $j = 1, 2$ where

$$H_1(r, \mathbf{x}) = \left(\frac{1 - \lambda(r)}{M^2 - r^2} + \frac{\lambda(r)}{w \circ \arg(\mathbf{x})^2 - r^2}\right)x_1$$

and

$$H_2(r, \mathbf{x}) = \left(\frac{1 - \lambda(r)}{M^2 - r^2} + \frac{\lambda(r)}{w \circ \arg(\mathbf{x})^2 - r^2} \right) x_2.$$

We observe first

$$\frac{\partial \psi_1}{\partial x_1} = \frac{\partial H_1}{\partial r} \, \frac{x_1}{|\mathbf{x}|} + \frac{\partial H_1}{\partial x_1}.$$

Calculating the suggested derivatives of H_1:

$$\begin{aligned} \frac{\partial H_1}{\partial r} = \lambda'(r) &\left(\frac{1}{w \circ \arg(\mathbf{x})^2 - r^2} - \frac{1}{M^2 - r^2} \right) x_1 \\ &+ 2r \left(\frac{1 - \lambda(r)}{(M^2 - r^2)^2} + \frac{\lambda(r)}{(w \circ \arg(\mathbf{x})^2 - r^2)^2} \right) x_1. \end{aligned}$$

$$\begin{aligned} \frac{\partial H_1}{\partial x_1} &= \frac{1 - \lambda(r)}{M^2 - r^2} + \frac{\lambda(r)}{w \circ \arg(\mathbf{x})^2 - r^2} \\ &\qquad - \frac{2\lambda(r) w \circ \arg(\mathbf{x})}{(w \circ \arg(\mathbf{x})^2 - r^2)^2} \, \frac{\partial}{\partial x_1} [w \circ \arg(\mathbf{x})] \; x_1 \\ &= \frac{1 - \lambda(r)}{M^2 - r^2} + \frac{\lambda(r)}{w \circ \arg(\mathbf{x})^2 - r^2} \\ &\qquad - \frac{2\lambda(r) w \circ \arg(\mathbf{x})}{(w \circ \arg(\mathbf{x})^2 - r^2)^2} \; w' \circ \arg(\mathbf{x}) \; \frac{x_1 x_2}{|\mathbf{x}|^2}. \end{aligned}$$

The last expression comes from the differentiation formula for radial functions. We see from these expressions that if $0 < x_1 < w$ there holds

$$\begin{aligned} \frac{\partial H_1}{\partial r}(x_1, x_1, 0) = \lambda'(x_1) &\left(\frac{1}{w(0)^2 - x_1^2} - \frac{1}{M^2 - x_1^2} \right) x_1 \\ &+ 2 \left(\frac{1 - \lambda(x_1)}{(M^2 - x_1^2)^2} + \frac{\lambda(x_1)}{(w(0)^2 - x_1^2)^2} \right) x_1^2, \end{aligned}$$

$$\frac{\partial H_1}{\partial x_1}(x_1, x_1, 0) = \frac{1 - \lambda(x_1)}{M^2 - x_1^2} + \frac{\lambda(x_1)}{w(0)^2 - x_1^2},$$

and

$$\begin{aligned} \frac{\partial \psi_1}{\partial x_1}(x_1, 0) = &\frac{1 - \lambda(x_1)}{M^2 - x_1^2} + \frac{\lambda(x_1)}{w(0)^2 - x_1^2} \\ &\lambda'(x_1) \left(\frac{1}{w(0)^2 - x_1^2} - \frac{1}{M^2 - x_1^2} \right) x_1 \\ &+ 2 \left(\frac{1 - \lambda(x_1)}{(M^2 - x_1^2)^2} + \frac{\lambda(x_1)}{(w(0)^2 - x_1^2)^2} \right) x_1^2. \end{aligned} \tag{2.28}$$

We turn next to $\partial\psi_2/\partial x_1$. It may be noted first that

$$\frac{\partial\psi_2}{\partial x_1} = \frac{\partial H_2}{\partial r}\,\frac{x_1}{|\mathbf{x}|} + \frac{\partial H_2}{\partial x_1}.$$

The derivative $\partial H_2/\partial r$ follows the same pattern as $\partial H_1/\partial r$ computed above:

$$\begin{aligned}\frac{\partial H_2}{\partial r} = \lambda'(r)\left(\frac{1}{w\circ\arg(\mathbf{x})^2 - r^2} - \frac{1}{M^2 - r^2}\right) x_2 \\ + 2r\left(\frac{1-\lambda(r)}{M^2 - r^2} + \frac{\lambda(r)}{w\circ\arg(\mathbf{x})^2 - r^2}\right) x_2.\end{aligned}$$

We see immediately then that for $0 < x_1 < w$

$$\frac{\partial H_2}{\partial r}(x_1, x_1, 0) = 0.$$

Furthermore, the derivative $\partial H_2/\partial x_1$ is given by

$$\frac{\partial H_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{1-\lambda(r)}{M^2 - r^2} + \frac{\lambda(r)}{w\circ\arg(\mathbf{x})^2 - r^2}\right)\; x_2.$$

From this we see that for $0 < x_1 < w$

$$\frac{\partial\psi_2}{\partial x_1}(x_1, 0) = \frac{\partial H_2}{\partial x_1}(x_1, x_1, 0) = 0.$$

Incidentally, the last value is geometrically evident since $\psi_2(x_1, 0) \equiv 0$.

So far the above computations tell us

$$\begin{aligned}\det D\psi(x_1, 0) &= \frac{\partial\psi_1}{\partial x_1}(x_1, 0)\frac{\partial\psi_2}{\partial x_2}(x_1, 0) - \frac{\partial\psi_2}{\partial x_1}(x_1, 0)\frac{\partial\psi_1}{\partial x_2}(x_1, 0) \\ &= \frac{\partial\psi_1}{\partial x_1}(x_1, 0)\frac{\partial\psi_2}{\partial x_2}(x_1, 0),\end{aligned}$$

and there is no need to calculate $\partial\psi_1/\partial x_2$. We turn to

$$\frac{\partial\psi_2}{\partial x_2} = \frac{\partial H_2}{\partial r}\,\frac{x_2}{|\mathbf{x}|} + \frac{\partial H_2}{\partial x_2}.$$

Here we only need to calculate

$$\begin{aligned}\frac{\partial H_2}{\partial x_2} = \frac{\partial}{\partial x_2}\left(\frac{1-\lambda(r)}{M^2 - r^2} + \frac{\lambda(r)}{w\circ\arg(\mathbf{x})^2 - r^2}\right)\; x_2 \\ + \frac{1-\lambda(r)}{M^2 - r^2} + \frac{\lambda(r)}{w\circ\arg(\mathbf{x})^2 - r^2}\end{aligned}$$

from which we see that when $0 < x_1 < w$

$$\frac{\partial H_2}{\partial x_2}(x_1, x_1, 0) = \frac{1 - \lambda(x_1)}{M^2 - x_1^2} + \frac{\lambda(x_1)}{w(0)^2 - x_1^2} > 0.$$

The positivity here follows because this is a convex combination of the positive numbers $g(x_1)/x_1$ and $h(x_1)/x_1$ at $\theta = 0$. Thus, in order to show $\det D\psi(x_1, 0) > 0$ it is only necessary to show the expression for $\partial\psi_1(x_1, 0)/\partial x_1$ given in (2.28) is positive. There are three lines in the display (2.28) which we can think of as three terms. The first term is precisely the convex combination of $g(x_1)/x_1$ and $h(x_1)/x_1$ just considered and is thus positive. The last term also is a convex combination of two positive numbers which happen to be $2[g(x_1)]^2$ and $2[h(x_1)]^2$. Finally, the middle term is $\lambda'(x_1)[h(x_1) - g(x_1)]$. Since $\lambda' \geq 0$, and $h(x_1) \geq g(x_1)$ for $0 < x_1 < w$, this term is nonnegative, and the assertion

$$\det D\psi(x_1, 0) > 0 \qquad \text{for} \qquad 0 \leq x_1 < w = w(0)$$

is established.

In order to deal with $D\psi(q)$ for an arbitrary point $q \in W$, or specifically to show $\det D\psi(q) > 0$, we apply a linear rotation as a change of variables. Specifically, setting $\arg(q) = \Theta$, we consider $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ given by counterclockwise rotation through the angle Θ, and the function $\rho^{-1} \circ \psi \circ \rho$ in particular. See Figure 2.5.

Recall from (2.27) that $\psi(\mathbf{x})$ is given by a scalar multiple of $\mathbf{x}$:

$$\psi(\mathbf{x}) = \left(\frac{1 - \lambda(|\mathbf{x}|)}{M^2 - |\mathbf{x}|^2} + \frac{\lambda(|\mathbf{x}|)}{w \circ \arg(\mathbf{x})^2 - |\mathbf{x}|^2} \right) \mathbf{x}. \tag{2.29}$$

Taking for variables $\xi = (\xi_1, \xi_2)$ in the domain of the composition, we note $w \circ \arg(\rho(\xi)) = w(\arg(\xi) + \Theta)$. This probably doesn't deserve a long and careful explanation, but I'm going to attempt to give one anyway.

The rotation ρ is of course linear, and it is traditional to express the values $\rho(\xi) = \rho(\xi_1, \xi_2)$ in terms of multiplication using the (constant) matrix

$$D\rho = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix}.$$

Since we are using row vectors to represent points like $(\xi_1, \xi_2) \in \rho^{-1}(W)$, writing down the full expression very carefully requires the use of transposes:

$$\rho(\xi) = \rho(\xi_1, \xi_2) = (D\rho\xi^T)^T.$$

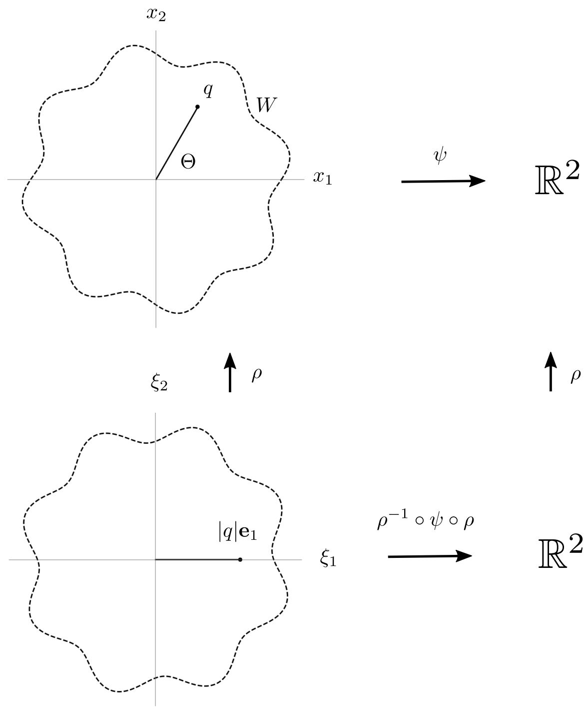

Figure 2.5: Mapping a star-shaped domain diffeomorphically to $\mathbb{R}^2$. Change of variables with a rotation $\rho : \rho^{-1}(W) \to W$ for application of the inverse function theorem.

On the other hand, $\xi = |\xi|(\cos \arg(\xi), \sin \arg(\xi))$, so

$$(D\rho\xi^T) = |\xi| \begin{pmatrix} \cos\Theta \cos\arg(\xi) - \sin\Theta \sin\arg(\xi) \\ \sin\Theta \cos\arg(\xi) + \cos\Theta \sin\arg(\xi) \end{pmatrix}$$

so

$$(D\rho\xi^T)^T = |\xi| \Big(\cos(\Theta + \arg(\xi)), \ \sin(\Theta + \arg(\xi)) \Big).$$

This doesn't mean $\arg \circ \rho(\xi) = \arg(\xi) + \Theta$, but it does mean there is some $j \in \mathbb{Z}$ for which

$$\arg \circ \rho(\xi) = \arg(\xi) + \Theta + 2j\pi,$$

and w is 2π periodic, so indeed $w \circ \arg(\rho(\xi)) = w(\arg(\xi) + \Theta)$ as noted.

In particular, $w(\Theta) = w \circ \arg \circ \rho(|q|\mathbf{e}_1)$. Also, $|\xi| = |\rho(\xi)|$, so making these substitutions we find

$$\begin{aligned}\rho^{-1}\circ\psi\circ\rho(\xi) &= \rho^{-1}\left[\left(\frac{1-\lambda(|\xi|)}{M^2-|\xi|^2}+\frac{\lambda(|\xi|)}{w\circ\arg\circ\rho(\xi)^2-|\xi|^2}\right)\rho(\xi)\right]\\ &= \left(\frac{1-\lambda(|\xi|)}{M^2-|\xi|^2}+\frac{\lambda(|\xi|)}{w(\arg(\xi)+\Theta)^2-|\xi|^2}\right)\xi. \qquad (2.30)\end{aligned}$$

Thus, we see the composition is given by a formula almost the same as the formula for ψ given in (2.29) and (2.27), and

$$D(\rho^{-1}\circ\psi\circ\rho) = D\rho^{-1}\ D\psi\circ\rho\ D\rho$$

so

$$\det D\psi\circ\rho = \det D(\rho^{-1}\circ\psi\circ\rho).$$

Evaluating at $\xi = \rho^{-1}(q) = (|q|\mathbf{e}_1)$ we obtain a formula

$$\det D\psi(q) = \det D(\rho^{-1}\circ\psi\circ\rho)(\rho^{-1}(q))$$

for the desired quantity.

The only difference in the formula is the additive constant in the expression

$$w(\arg(\xi)+\Theta)$$

in (2.30) in comparison to

$$w\circ\arg(\mathbf{x})$$

in (2.29). The question is: What difference does this make in the calculation of $D(\rho^{-1}\circ\psi\circ\rho)$ in comparison to the calculation of $D\psi$?

Looking back at the previous calculation we should now take

$$H_1(r,\xi) = \left(\frac{1-\lambda(r)}{M^2-r^2}+\frac{\lambda(r)}{w(\arg(\xi)+\Theta)^2-r^2}\right)\xi_1.$$

The calculation $\partial H_1/\partial r$ follows the same pattern with only a simple and expected substitution with

$$\begin{aligned}\frac{\partial H_1}{\partial r} &= \lambda'(r)\left(\frac{1}{w(\arg(\xi)+\Theta)^2-r^2}-\frac{1}{M^2-r^2}\right)x_1\\ &\qquad + 2r\left(\frac{1-\lambda(r)}{(M^2-r^2)^2}+\frac{\lambda(r)}{(w(\arg(\xi)+\theta)^2-r^2)^2}\right)x_1\end{aligned}$$

and

$$\frac{\partial H_1}{\partial r}(|q|,|q|,0) = \lambda'(|q|)\left(\frac{1}{w(\Theta)^2-|q|^2} - \frac{1}{M^2-|q|^2}\right)|q| \\ + 2\left(\frac{1-\lambda(|q|)}{(M^2-|q|^2)^2} + \frac{\lambda(|q|)}{(w(\Theta)^2-|q|^2)^2}\right)|q|^2.$$

The first calculation where the additive constant requires some consideration is in the calculation of $\partial H_1/\partial \xi_1$. In this case

$$\frac{\partial H_1}{\partial \xi_1} = \frac{1-\lambda(r)}{M^2-r^2} + \frac{\lambda(r)}{w\circ\arg(\mathbf{x})^2-r^2} \\ - \frac{2\lambda(r)w(\arg(\xi)+\Theta)}{(w(\arg(\xi)+\Theta)^2-r^2)^2}\ \frac{\partial}{\partial \xi_1}[w(\arg(\xi)+\Theta)]\ \xi_1. \quad (2.31)$$

Thus, we need to calculate

$$\frac{\partial}{\partial \xi_1}[w(\arg(\xi)+\Theta)].$$

Observe however that $\tilde{w}:\mathbb{R}\to\mathbb{R}$ by $\tilde{w}(\theta) = w(\theta+\Theta)$ is a radial function with $\tilde{w}'(\theta) = w'(\theta+\Theta)$. Therefore,

$$\begin{aligned} \frac{\partial}{\partial \xi_1}[w(\arg(\xi)+\Theta)] &= \frac{\partial}{\partial \xi_1}[\tilde{w}\circ\arg(\xi)] \\ &= -\tilde{w}'\circ\arg(\xi)\ \frac{\xi_2}{|\xi|} \\ &= -w'(\arg(\xi)+\Theta)\ \frac{\xi_2}{|\xi|}. \end{aligned}$$

We see from this that the expression in (2.31) takes the form

$$\frac{\partial H_1}{\partial \xi_1} = \frac{1-\lambda(r)}{M^2-r^2} + \frac{\lambda(r)}{w(\arg(\xi)+\Theta)^2-r^2} \\ - \frac{2\lambda(r)w(\arg(\xi)+\Theta)}{(w(\arg(\xi)+\Theta)^2-r^2)^2}\ w'(\arg(\xi)+\Theta)\ \frac{\xi_1\xi_2}{|\xi|^2}.$$

Evaluating at $\rho^{-1}(q) = (|q|,0)$

$$\frac{\partial H_1}{\partial \xi_1}(|q|,|q|,0) = \frac{1-\lambda(|q|)}{M^2-|q|^2} + \frac{\lambda(|q|)}{w(\Theta)^2-|q|^2},$$

and setting $\Psi = (\Psi_1, \Psi_2) = \rho^{-1} \circ \psi \circ \rho$,

$$\begin{aligned}\frac{\partial \Psi_1}{\partial \xi_1}(|q|, 0) = \frac{1-\lambda(|q|)}{M^2-|q|^2} + \frac{\lambda(|q|)}{w(\Theta)^2-|q|^2} \\ \lambda'(|q|)\left(\frac{1}{w(\Theta)^2-|q|^2} - \frac{1}{M^2-|q|^2}\right)\xi_1 \\ + 2\left(\frac{1-\lambda(|q|)}{(M^2-|q|^2)^2} + \frac{\lambda(|q|)}{(w(\Theta)^2-|q|^2)^2}\right)|q|^2 \\ > 0\end{aligned}$$

much as in the calculation of $D\psi(x_1, 0)$. Following the previous calculation, we now use

$$H_2(r, \xi) = \left(\frac{1-\lambda(r)}{M^2-r^2} + \frac{\lambda(r)}{w(\arg(\mathbf{x})+\Theta)^2-r^2}\right)\xi_2$$

and essentially the same calculation gives

$$\frac{\partial \Psi_2}{\partial \xi_1}(|q|, 0) = \frac{\partial H_2}{\partial \xi_1}(|q|, |q|, 0) = 0.$$

Again this is geometrically evident since $\Psi_2(\xi_1, 0) \equiv 0$.

We have then

$$\det D\Psi(|q|, 0) = \frac{\partial \Psi_1}{\partial \xi_1}(|q|, 0)\frac{\partial \Psi_2}{\partial \xi_2}(|q|, 0),$$

and it remains to show $\partial \Psi_2(|q|, 0)/\partial \xi_2 > 0$. In fact,

$$\frac{\partial \Psi_2}{\partial \xi_2}(|q|, 0) = \frac{\partial H_2}{\partial \xi_2}(|q|, |q|, 0).$$

and noting the formula

$$\frac{\partial}{\partial \xi_2}[w(\arg(\xi)+\Theta)] = w'(\arg(\xi)+\Theta)\ \frac{\xi_1}{|\xi|}, \tag{2.32}$$

$$\begin{aligned}\frac{\partial H_2}{\partial \xi_2} = \frac{\partial}{\partial \xi_2}\left(\frac{1-\lambda(r)}{M^2-r^2} + \frac{\lambda(r)}{w(\arg(\xi)+\Theta)^2-r^2}\right)\ \xi_2 \\ + \frac{1-\lambda(r)}{M^2-r^2} + \frac{\lambda(r)}{w(\arg(\xi)+\Theta)^2-r^2}\end{aligned}$$

so that
$$\frac{\partial H_2}{\partial \xi_2}(|q|, |q|, 0) = \frac{1-\lambda(|q|)}{M^2 - |q|^2} + \frac{\lambda(|q|)}{w(\Theta)^2 - |q|^2} > 0.$$
It follows that $\det D\psi(q) > 0$ for all $q \in W$ as claimed.

Exercise 2.16. Give a justification for the formula (2.32). Be careful not to differentiate the argument function because the argument function is not even continuous in general.

The proof of Lemma 2.1 is now completed as follows: For any $\mathbf{y} \in \mathbb{R}^2$ we have shown
$$\det D\psi(\psi^{-1}(\mathbf{y})) > 0.$$
By the inverse function theorem the unique inverse $\psi^{-1} : \mathbb{R}^2 \to W$ of the C^k function $\psi : W \to \mathbb{R}^2$ is given locally near $\mathbf{y}$ by the inverse of a C^k diffeomorphism
$$\psi\big|_{U_0} : U_0 \to \psi(U_0)$$
where U_0 is an open set with $q = \psi^{-1}(\mathbf{y}) \in\in U_0 \subset W$ so that $\mathbf{y}$ is in the open set $\psi(U_0)$. Consequently, $\psi^{-1} \in C^k(\psi(U_0))$, and since $\mathbf{y} \in \mathbb{R}^2$ is arbitrary, $\psi^{-1} \in C^k(\mathbb{R}^2)$. □

Exercise 2.17. Give a proof of Lemma 2.1 by replacing the function $(1-\lambda)g + \lambda h$ with...

Exercise 2.18. (Whitney extension theorem)

We are now in a good position to show a graph is a surface. We need only incorporate a translation: Recall that for $p = (\mathbf{x}, f(\mathbf{x})) \in \mathcal{G}$ we have taken $\epsilon > 0$ for which $\mathcal{G} \cap B_\epsilon$ is precisely $\{(\xi(p) + a(\mathbf{v})\mathbf{v}, f(\xi(p) + a\mathbf{v})) : \mathbf{v} \in \mathbb{S}^1\}$ for some $a \in C^0(\mathbb{S}^1)$ for which we have establshed various properties including the relation $a(\mathbf{v}) = w(\theta) = w \circ \arg(\mathbf{v})$ where $w \in C^k(\mathbb{R})$. Letting $\psi : W \to \mathbb{R}^2$ be a C^k diffeomorphism, we define the function $X : \mathbb{R}^2 \to \mathbb{R}^3$ by
$$X(\mathbf{u}) = (\xi(p) + \psi^{-1}(\mathbf{u}), f(\xi(p) + \psi^{-1}(\mathbf{u}))).$$
The function X is in $C^k(\mathbb{R}^2 \to \mathbb{R}^3)$. In particular, X is continuous. Furthermore, we have shown
$$\begin{aligned}\mathcal{G} \cap B_\epsilon(p) &= \left\{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \{\xi(p) + r\mathbf{v} : \mathbf{v} \in \mathbb{S}^1\}\right\} \\ &= \left\{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \{\xi(p) + \mathbf{w} : \mathbf{w} \in W\}\right\}.\end{aligned}$$

Since $W = \psi^{-1}(\mathbb{R}^2)$, this means

$$\mathcal{G} \cap B_\epsilon(p) = \left\{ (\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \{\xi(p) + \psi^{-1}(\mathbf{u}) : \mathbf{u} \in \mathbb{R}^2\} \right\}.$$

That is $\mathcal{G} \cap B_\epsilon(p) = X(\mathbb{R}^2)$ as required by Definition 5.

In fact, since the projection $\pi : \mathcal{G} \cap B_\epsilon(p) \to \mathbb{R}^2$ by

$$\pi(\mathbf{x}, f(\mathbf{x})) = \mathbf{x} \in \{\xi(p) + \mathbf{w} : \mathbf{w} \in W\}$$

is one-to-one and continuous, the mapping $\eta : \mathcal{G} \cap B_\epsilon(p) \to \mathbb{R}^2$ by

$$\eta(\mathbf{x}, f(\mathbf{x})) = \psi(\pi(\mathbf{x}, f(\mathbf{x})) - \xi(p)) = \psi(\mathbf{x} - \xi(p))$$

is a continuous inverse of X. The mapping $X : \mathbb{R}^2 \to \mathcal{G} \cap B_\epsilon(p)$ is a homeomorphism as required by condition **(S1)**.

Finally, we consider $dX_{\mathbf{u}} : \mathbb{R}^2 \to \mathbb{R}^3$. Let $\psi^{-1} = \mathbf{w} = (w_1, w_2)$ with $w_1 = w_1(\mathbf{u})$ and $w_2 = w_2(\mathbf{u})$. In terms of matrix multiplication

$$dX_{\mathbf{u}}(\mathbf{v}) = \left[\begin{pmatrix} \dfrac{\partial w_1}{\partial u_1} & \dfrac{\partial w_1}{\partial u_2} \\ \dfrac{\partial w_2}{\partial u_1} & \dfrac{\partial w_2}{\partial u_2} \\ \dfrac{\partial}{\partial u_1} f(\xi(p) + w) & \dfrac{\partial}{\partial u_2} f(\xi(p) + w) \end{pmatrix} \mathbf{v}^T \right]^T.$$

But the matrix

$$D\mathbf{w}(\mathbf{u}) = \begin{pmatrix} \dfrac{\partial w_1}{\partial u_1} & \dfrac{\partial w_1}{\partial u_2} \\ \dfrac{\partial w_2}{\partial u_1} & \dfrac{\partial w_2}{\partial u_2} \end{pmatrix} = [D\psi \circ \psi^{-1}(\mathbf{u})]^{-1}$$

has rank 2, so the matrix associated with $dX_{\mathbf{u}}$ has rank 2 as well, and this completes the proof that a graph is a surface according to Definition 5.

Exercise 2.19. Verify directly (with a simplified argument) that for each $r > 0$ and $q \in \mathbb{R}^3$ the set

$$\partial B_r(q) = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x} - q| = r\}$$

is an embedded surface according to Definition 5.

2.4.1 Local graphs

We have shown that a graph is a surface. We now show that every surface is locally a graph near each point.

Exercise 2.20. Verify for any $r > 0$ and $q \in \mathbb{R}^3$ that $\partial B_r(q)$ is a surface. (This is at the moment a repeat of Exercise 2.19.)

Exercise 2.21. Verify for any $r > 0$ and $p = (p_1, p_2) \in \mathbb{R}^2$ that $\mathcal{C} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1 - p_1)^2 + (x_2 - p_2)^2 = r^2\}$ is a surface.

2.5 More Alternative Definitions

This section may also be called Gallot, Hulin, and Lafontaine, or perhaps French differential geometry.

Definition 6. A set $\mathcal{S} \subset \mathbb{R}^3$ is said to be a **Lafontaine surface** if for each $p \in \mathcal{S}$, there exists some open set $V \subset \mathbb{R}^3$ with $p \in V$ and an open set $U \subset \mathbb{R}^2$ with a local parameterization $X \in C^k(U \to \mathbb{R}^3)$ satisfying $X(U) = V \cap \mathcal{S}$.

Recall that by a **local parameterization** of this sort we mean $X : U \to V \cap \mathcal{S}$ is a homeomorphism and $dX_{\mathbf{u}} : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one. This is precisely just the permissive definition of a surface given in Definition 2. There is nothing new here, but the new name is nice. Also, it fits a certain template to refer to local parameterizations of the permissive variety as **Lafontaine parameterizations**.

Definition 7. (Hulin diffeomorphism) A set $\mathcal{S} \subset \mathbb{R}^3$ is a **Hulin surface** if for each $p \in \mathcal{S}$ there exists an open set $V \subset \mathbb{R}^3$ with $p \in V$ and an open set $Z \subset \mathbb{R}^3$ with $\mathbf{0} = (0,0,0) \in Z$ and a C^k diffeomorphism $\Xi : V \to Z$ for which

$$\Xi(V \cap \mathcal{S}) = Z \cap (\mathbb{R}^3 \times \{0\}). \tag{2.33}$$

The set $Z \cap (\mathbb{R}^3 \times \{0\})$ can be written in various forms. All of them seem to be a little cumbersome or complicated, but the idea is simple. For the sake of brevity we sometimes use the notation

$$\begin{aligned} \underline{Z} &= \{(u_1, u_2, u_3) \in Z : u_3 = 0\} \\ &= \{(u_1, u_2, 0) : \mathbf{u} = (u_1, u_2, 0) \in Z\}. \end{aligned}$$

Naturally the diffeomorphism $\Xi \in C^k(V \to Z)$ in Defn 7 is called a **Hulin diffeomorphism**.

Definition 8. (gallows function or gallows submersion[5]) A set $\mathcal{S} \subset \mathbb{R}^3$ is a **Gallot surface** if for each $p \in \mathcal{S}$ there exists an open set $V \subset \mathbb{R}^3$ with $p \in V$ and a (real valued) function $g \in C^k(V)$ such that

$$\mathbf{0} \notin Dg(V) = \{Dg(\mathbf{x}) : \mathbf{x} \in V\} \tag{2.34}$$

and

$$V \cap \mathcal{S} = g^{-1}(0) = \{\mathbf{x} \in V : g(\mathbf{x}) = 0\}. \tag{2.35}$$

The condition (2.34) is what makes g a submersion.[6] A function $g \in C^k(V)$ satisfying (2.34) and (2.35) shall be called a local **gallows function** for the surface $\mathcal{S}$.

One thing all these definitions have in common is the involvement of an open set $V \subset \mathbb{R}^3$ with $p \in V$. That is not all:

Lemma 2.2. (Gallot, Hulin, Lafontaine lemma) Let p be a point in a C^k embedded surface $\mathcal{S} \subset \mathbb{R}^3$.

(Gallot) If V is an open set in $\mathbb{R}^3$ with $p \in V$ and $g \in C^k(V)$ is a gallows function, then

$$\tilde{g} = g\big|_{\tilde{V}} \in C^k(\tilde{V})$$

is a gallows function for every open set $\tilde{V} \subset \mathbb{R}^3$ with $p \in \tilde{V} \subset V$.

(Hulin) If V is an open set in $\mathbb{R}^3$ with $p \in V$ and $\Xi \in C^k(V \to Z)$ is a Hulin diffeomorphism, then

$$\tilde{\Xi} = \Xi\big|_{\tilde{V}} \in C^k(V \to \tilde{Z})$$

where $\tilde{Z} = \Xi(\tilde{V})$ is a Hulin diffeomorphism for every open set $\tilde{V} \subset \mathbb{R}^3$ with $p \in \tilde{V} \subset V$.

(Lafontaine) V is an open set in $\mathbb{R}^3$ with $p \in V$ and $X \in C^k(U \to \mathbb{R}^3)$ is a Lafontaine parameterization, then

$$\tilde{X} = X\big|_{\tilde{V}} \in C^k(\tilde{U} \to \mathbb{R}^3)$$

where $\tilde{U} = \xi(\tilde{V})$ is a Lafontaine parameterization for every open set $\tilde{V} \subset \mathbb{R}^3$ with $p \in \tilde{V} \subset V$.

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[5] Isn't **submersion** a cool word?

[6] More generally, if $f : V \to \mathbb{R}^m$ with $V \subset \mathbb{R}^m$ and $n < m$, then $f \in C^1(V \to \mathbb{R}^m)$ is a **submersion** if $df_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}^m$ is surjective for each $\mathbf{x} \in V$.

It follows from Lemma 2.2 that if p is a point in a surface $\mathcal{S}$ and V_1, V_2 and V_3 are open subsets of $\mathbb{R}^3$ for which $p \in V_1 \cap V_2 \cap V_3$ and

(i) $g \in C^k(V_1)$ is a gallows function,

(ii) $\Xi \in C^k(V_2 \to Z_2)$ is a Hulin diffeomorphism, and

(iii) $X \in C^k(U_2 \to \mathbb{R}^3$ is a Lafontaine parameterization,

then there exist all three things, a gallows function, a Hulin diffeomorphism, and a Lafontaine parameterization all based on the same set $V = V_1 \cap V_2 \cap V_3$ at $p \in \mathcal{S}$. In fact, one doesn't even need to assume **(i)** and **(ii)** for this conclusion.

Lemma 2.3. (Gallot, Hulin, Lafontaine proposition) If $\mathcal{S}$ is a C^k embedded surface and $p \in \mathcal{S}$, then there exists an open set $V \subset \mathbb{R}^3$ with $p \in V$ and the following hold:

(Gallot) There exists a function $g \in C^k(V)$ such that

$$\mathbf{0} \notin Dg(V) \qquad \text{and} \qquad V \cap \mathcal{S} = g^{-1}(0).$$

(Hulin) There exists an open set $Z \subset \mathbb{R}^3$ with $\mathbf{0} \in Z$ and a C^k diffeomorphism $\Xi : V \to Z$ such that

$$\Xi[V \cap \mathcal{S}) = \underline{Z} = Z \cap (\mathbb{R}^3 \times \{0\}).$$

(Lafontaine) There exists an open set $U \subset \mathbb{R}^2$ and a local parameterization $X \in C^k(U \to \mathbb{R}^3$ with

$$X(U) = \mathcal{S} \cap V.$$

Proof: As mentioned above Lafontaine is just the permissive definition of a C^k embedded surface, so we can start with a local parameterization $X_1 : U_1 \to \mathbb{R}^3$.

Chapter 3

Symmetric Surfaces

Let $\mathbf{n} \in \mathbb{S}^2$, and consider the plane

$$P = P(\mathbf{n}) = P_a(\mathbf{n}) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{n} = a\}.$$

A surface $\mathcal{S} \subset \mathbb{R}^3$ is said to be symmetric with respect to P if

$$X + 2(a - X \cdot \mathbf{n})\mathbf{n} \in \mathcal{S} \qquad \text{whenever} \qquad X \in \mathcal{S}.$$

Exercise 3.1. Show that if $\mathcal{S}$ is symmetric with respect to P, then

$$\{X + 2(a - X \cdot \mathbf{n})\mathbf{n} : X \in \mathcal{S}\} = \mathcal{S}.$$

Consider a symmetric surface $\mathcal{S}$ with a point $p \in \mathcal{S} \cap P$ where $P = P_a(\mathbf{n})$ is the plane of symmetry. Given a local parameterization

$$X \in C^k(B_1(\mathbf{0}) \to \mathbb{R}^3)$$

If $\mathcal{S}$ connected and symmetric with respect to P, then $\mathcal{S}$ must intersect P, and either

(i) $\mathcal{S} \subset P$ or

(ii) At each point $p_0 \in \mathcal{S} \cap P$, the surface $\mathcal{S}$ is orthogonal to P, that is, $N_0 \cdot \mathbf{n} = 0$.

If $\mathcal{S}$ is compact and symmetric with respect to $P = P_a(\mathbf{n})$, then $a = a(\mathbf{n})$ is uniquely determined, and at each $p_0 \in \mathcal{S} \cap P$ (if there are any such points) $N_0 \cdot \mathbf{n} = 0$.

Part II

Appendix

Appendix A

An interesting geometric rate

When I first made the computation (2.22) and got the value 1 my (misguided) geometric intuition suggested to me there must have been an error in the computation or that there must be something I was not understanding correctly. Fortunately the computation wasn't so long or involved, and after checking it a dozen or so times, and some times very carefully, I came to the belief that the answer must somehow be correct, but I still didn't understand it geometrically or more precisely I was under the delusion that I understood something geometrically suggesting that answer could not be correct. After contemplating the computation for a while longer I realized there was indeed something I was not understanding correctly, and I envisioned the illustration below which it seemed to me made the point clear in my mind, and I thought the point was an interesting one.

Say you have a differentiable function of one variable $f : (0, \infty) \to \mathbb{R}$, and at some $x_0 > 0$ the function takes the value p_3. Now imagine you calculate the distance R from each point $(x, f(x))$ on the graph of f to the point $(0, p_3)$. Roughly speaking one expects the rate of change of this distance

$$\frac{dR}{dx}$$

to depend on the rate of change of the horizontal distance

$$\frac{d}{dx}(x - 0) = 1 \tag{A.1}$$

and the rate of change of the vertical distance

$$\frac{d}{dx}(f(x) - p_3) = f'(x), \tag{A.2}$$

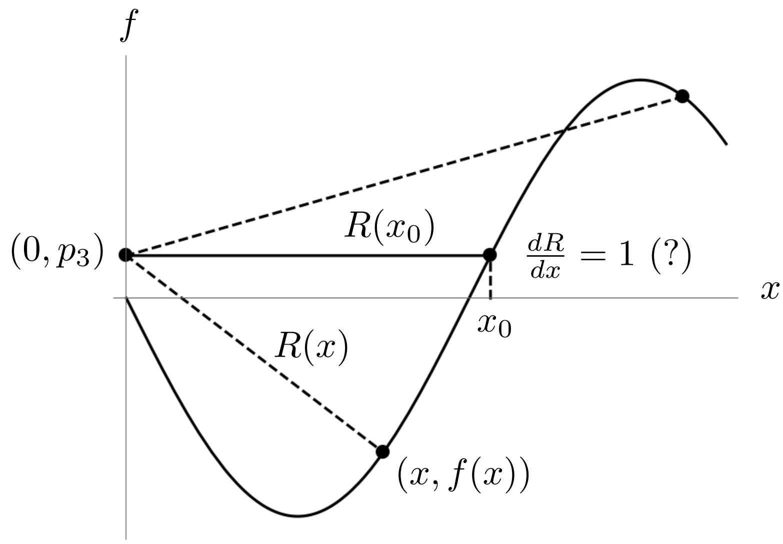

Figure A.1: The distance from the graph of a function to a fixed point as a function of x.

and one might mistakenly think this means dR/dx must be at least 1 and always greater than 1 unless $f'(x) = 0$.

Turning specifically to the point $(x_0, f(x_0)) = (x_0, p_3)$, one might expect for example that as f increases through the value $f(x_0) = p_3$ as indicated in Figure A.1 there must hold $dR/dx(x_0) > 1$. After all the **rate of change** of the vertical distance should play a role, and the fact that the actual value of the vertical displacement $f(x_0) - p_3$ happens to vanish at this point should be more or less irrelevant.

Notice the suggested intuition is at least partially correct as x increases to x_0. At the radius labeled $R(x)$ in the figure there are indeed (nonzero) contributions from the two terms (A.1) and (A.2) and as a result of the contribution of (A.2) in particular the value of dR/dx may be strictly greater than 1.

On the other hand, one can start to see there is something fishy here because in this case the quantity $f(x) - p_3$ is negative and the quantity in (A.2) is positive which means that in terms of distance, the real vertical **distance** is $p_3 - f(x)$ and that positive quantity is decreasing.

This is sort of the key. While it is true that f is increasing and the vertical "distance" $f(x) - p_3$ is changing at a nonzero rate as x passes through x_0 just like at other points, one is transitioning from a situation in which the actual vertical distance $|f(x) - p_3|$ is decreasing and then increasing. Technically, setting aside the effect of the chain rule, the relevant quantity is captured by

$$\frac{1}{2}\frac{d}{dx}(f(x) - p_3)^2 = (f(x) - p_3)f'(x).$$

In this way, one sees the contribution from the change in vertical distance (represented by $f'(x)$) becomes irrelevant precisely when $f(x_0) = p_3$, while it is still the case that when $f'(x) = 0$ at any particular point, then the value of dR/dx is only effected by the change in the horizontal distance—at least out of the two terms given in (A.1) and (A.2). As it turns out, the value of dR/dx is also dependent in this case on the actual value $f(x) - p_3$, so things are just generally a little more complicated than the simple intuition above suggests.

Here is the actual (potentially confusing) computation:

$$R(x) = \sqrt{x^2 + (f(x) - p_3)^2}.$$

$$\frac{dR}{dx}(x) = \frac{x + (f(x) - p_3)f'(x)}{\sqrt{x^2 + (f(x) - p_3)^2}}.$$

Exercise A.1. Assuming f is increasing on an interval $(a, b) \subset (0, \infty)$ with $x_0 \in (a, b)$ and $f(x_0) = p_3$ as described above, describe geometrically all conditions under which dR/dx can satisfy

(i) $dR/dx < 1$.

(ii) $dR/dx < 0$.

Can it be the case, for example, that $dR/dx < 1$ when $x > x_0$?

Appendix B

Arguments, polar coordinates, and radial graphs

We define the real principal argument function $\arg = \arg_0 : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ by

$$\arg(x,y) = \begin{cases} \tan^{-1}(y/x), & x > 0,\ y \geq 0 \\ \pi/2 + \tan^{-1}(-x/y), & x \leq 0,\ y > 0 \\ \pi + \tan^{-1}(y/x), & x < 0,\ y \leq 0 \\ 3\pi/2 + \tan^{-1}(-x/y), & x \geq 0\ y < 0. \end{cases} \tag{B.1}$$

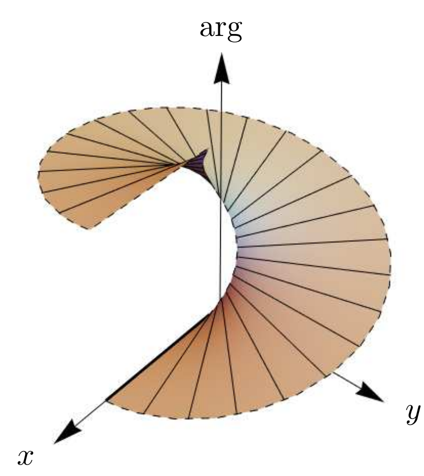

Figure B.1: The radial graph associated with the principal argument.

This argument, as we have defined it, has a singularity along the ray $\{(x,0) : x > 0\}$, however, $\arg \in C^\infty(\mathbb{R}^2\backslash\{(x,0) : x \geq 0\}$. This regularity, though confirmed by the illustration in Figure B.1, may appear nominally at odds with the complicated piecewise formula (B.1). The regularity across the boundaries indicated in (B.1) may be confirmed by the alternative form

$$\arg(x,y) = \begin{cases} \tan^{-1}(y/x), & x > 0,\ y \geq 0 \\ \cot^{-1}(x/y), & y > 0 \\ \pi + \tan^{-1}(y/x), & x < 0 \\ \pi + \cot^{-1}(x/y), & y < 0 \end{cases} \tag{B.2}$$

with overlapping regions. That each of these smooth formulas agree over their open overlaps follows from trigonometric formulas. For example,

$$\pi + \tan^{-1}(y/x) = \pi/2 + \tan^{-1}(-x/y)$$

for $x < 0$ and $y > 0$ because in this region $-\pi/2 < \tan^{-1}(y/x) < 0$ implying $0 < \pi/2 + \tan^{-1}(y/x) < \pi/2$, and

$$\tan(\pi/2 + \tan^{-1}(y/x)) = -\cot(\tan^{-1}(y/x)) = -x/y.$$

The main assertion/observation of this appendix is that given $k \in \mathbb{N}_0 = \{0,1,2,3,\ldots\}$ and a function $w \in C^k(\mathbb{R})$ satisfying the periodicity condition

$$w(\theta + 2\pi) = w(\theta), \qquad \theta \in \mathbb{R},$$

the function $f : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ with values

$$f(x,y) = w \circ \arg(x,y) \tag{B.3}$$

satisfies $f \in C^k(\mathbb{R}^2\backslash\{(0,0)\})$, and in the case $k \geq 1$

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2+y^2}\ w' \circ \arg(x,y) \tag{B.4}$$

and

$$\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}\ w' \circ \arg(x,y). \tag{B.5}$$

This assertion stands in some contrast to the lack of regularity in the argument arg itself so that a formula involving a composition with arg, as for example the one in (B.3), cannot be approached using a direct analysis of the composition along the singular ray $\{(x, 0) : x > 0\}$.

A function $f : \mathbb{R}^2 \backslash \{(0, 0)\} \to \mathbb{R}$ with values having the form $f(x, y) = w \circ \arg(x, y)$ for some periodic $w \in C^k(\mathbb{R})$ will be called a **radial function** with graph

$$\Big\{(x, y, f(x, y)) : (x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\}\Big\}$$

referred to as a C^k **radial graph**. It is convenient to extend this terminology to allow $f : U \to \mathbb{R}$ with U an open subset of $\mathbb{R}^2 \backslash \{(0, 0)\}$ and the periodic function $w \in C^k(I)$ for some appropriate interval $I \subset \mathbb{R}$. Perhaps the following is adequate:

Definition 9. Given $k \in \mathbb{N}_0$, and $f : U \to \mathbb{R}$ with U an open subset of $\mathbb{R}^2 \backslash \{(0, 0)\}$, the function f is said to be a C^k **radial function** and the set

$$\Big\{(x, y, f(x, y)) : (x, y) \in U\Big\}$$

is a C^k **radial graph** if the values of f have the form $f(x, y) = w \circ \arg(x, y)$ where

(i) $w \in C^k(I)$ for some open interval $I \subset \mathbb{R}$,

(ii) $w(\theta + 2\pi) = w(\theta)$ whenever $\theta + 2\pi, \theta \in I$, and

(iii) Given any $(x, y) \in U$, there is some $\delta > 0$ and a function $w_0 \in C^k(\mathbb{R})$ with

$$w\Big|_{(\arg(x,y)-\delta, \arg(x,y)+\delta)} = w_0\Big|_{(\arg(x,y)-\delta, \arg(x,y)+\delta)}.$$

Naturally, it makes sense to consider also C^∞ radial graphs taking simply $w \in C^\infty(I) = \cap_k C^k(I)$.

Exercise B.1. Show

$$\arg\Big|_{\mathbb{R}^2 \backslash \{(x,0): x \geq 0\}}$$

with arg defined as above is a C^∞ radial function.

If $U \subset \mathbb{R}^2 \backslash \{(x, 0) : x \geq 0\}$, then the assertion $w \in C^\infty(U)$ follows simply from the fact that $\arg \in C^\infty(\mathbb{R}^2 \backslash \{(x, 0) : x \geq 0\})$ and the chain and product rules. Formulas (B.4) and (B.5) hold in particular.

The remainder of this section offers an explanation for the regularity assertion(s) concerning radial functions, some further discussion of real argument functions, and some relations with the polar coordinates map. The brief introductory comments concerning radial graphs reduce the question to points on the singular ray $\{(x,0) : x > 0\}$, and the basic explanation of regularity along this ray can be made very simple. We take somewhat of a "scenic route" including some additional details along the way to the simple explanation.

Consider first continuity: If $(x,0)$ lies along the singular ray, then clearly

$$\lim_{\xi\to x,\ \eta\to 0^+} w\circ\arg(\xi,\eta) = w(0)$$

simply by the one-sided continuity of the principal argument. If (ξ,η) satisfies $\xi > 0$ and $\eta < 0$ on the other hand, then $3\pi/2 < \arg(\xi,\eta) < 2\pi$. In this case,

$$-\pi/2 < \arg(\xi,\eta) - 2\pi < 0$$

and $w\circ\arg(\xi,\eta) = w(\arg(\xi,\eta)-2\pi)$. Moreover, if (ξ,η) tends to $(x,0)$ with $\eta < 0$

$$\lim_{\xi\to x,\ \eta\to 0^-} \arg(\xi,\eta) = 2\pi$$

and the quantity $\arg(\xi,\eta) - 2\pi$ tends to 0. Thus,

$$\lim_{\xi\to x,\ \eta\to 0^-} w\circ\arg(\xi,\eta) = \lim_{\xi\to x,\ \eta\to 0^-} w(\arg(\xi,\eta)-2\pi) = w(0)$$

as well. This shows

$$\lim_{(\xi,\eta)\to(x,0)} w\circ\arg(\xi,\eta) = w(0)$$

and the composition $w\circ\arg$ is continuous on $\mathbb{R}^2\backslash\{(0,0)\}$ even though $\arg \notin C^0(\mathbb{R}^2\backslash\{(0,0)\})$.

Next consider $\partial\arg/\partial x : \mathbb{R}^2\backslash\{(0,0)\}$. Notice $\arg(x,0)\equiv 0$ for $x > 0$, so

$$\frac{\partial\arg}{\partial x}(x,y) = -\frac{y}{x^2+y^2}$$

even along the singular ray. Thus, while arg itself is not in $C^0(\mathbb{R}^2\backslash\{(0,0)\})$, there holds

$$\frac{\partial\arg}{\partial x} \in C^\infty(\mathbb{R}^2\backslash\{(0,0)\}).$$

The partial derivative $\partial \arg / \partial y$ of course does not exist along the singular ray, but

$$\frac{\partial \arg}{\partial y} \in C^\infty(\mathbb{R}^2 \backslash \{(x,0) : x \geq 0\})$$

as mentioned above with

$$\frac{\partial \arg}{\partial y}(x,y) = \frac{x}{x^2+y^2}.$$

We must be somewhat careful at this point: We cannot equate

$$\frac{\partial \arg}{\partial y}(x,0) \qquad \text{and} \qquad \frac{1}{x}$$

for $x > 0$, but these values do give a continuous extension of the function $\partial \arg / \partial y$ to a function in $C^\infty(\mathbb{R}^2 \backslash \{(0,0)\})$.

Recall we already know $f = w \circ \arg \in C^k(\mathbb{R}^2 \backslash \{(x,0) : x \geq 0\})$ when $w \in C^k(\mathbb{R})$ for some $k > 0$, so now we may focus on a point $(x,0)$ with $x > 0$.

The discussion of the principal argument function arg above was based on the real arctangent and arccotangent functions. We took $\tan^{-1}(x) = \theta$ to be the unique value θ with $-\pi/2 < \theta < \pi/2$ such that $\tan\theta = x$. Of course, there are other natural real branches of the arctangent which may be defined by $\tan_j^{-1}(x) = \tan^{-1}(x) + j\pi$ for $j \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$. Similarly, $\cot^{-1} : \mathbb{R} \to (0,\pi)$ is a decreasing C^∞ diffeomorphism, and there are also values

$$\cot_j^{-1}(x) = \cot^{-1}(x) + j\pi$$

for $j \in \mathbb{Z}$ giving a family of branches each member of which is a decreasing C^∞ diffeomorphism.

In a certain sense corresponding to the branches of $\cot^{-1}$ there are branches of the argument $\arg_j : \mathbb{R}^2 \backslash \{(0,0)\} \to [j\pi, (2+j)\pi)$ with values

$$\arg_j(x) = \arg(x) + j\pi$$

for $j \in \mathbb{Z}$. There is also an alternative argument function[1] loosely based on the branches of $\tan^{-1}$. Let us denote the principal branch of this function by

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[1]There are many alternative argument functions actually, but the two of them introduced here are probably the most common or popular.

$\text{crg} : \mathbb{R}^2\backslash\{(0,0)\} \to (-\pi, \pi]$ with

$$\text{crg}(x,y) = \begin{cases} \tan^{-1}(y/x), & x > 0 \\ \cot^{-1}(x/y), & y > 0 \\ \tan_1^{-1}(y/x), & x < 0 \\ \cot_{-1}^{-1}(x/y), & y < 0. \end{cases} \tag{B.6}$$

The function arg we have defined may be called the *positive* principal argument, and crg may be called[2] the *central* or *centered* principal argument. The introduction of the function $\text{crg} : \mathbb{R}^2\backslash\{(0,0)\} \to (-\pi, \pi]$ is the simple explanation for the regularity of a radial function $f = w \circ \arg : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ along the singular ray. We distinguish the (very simple) key observation with a relatively grandiose statement and proof as follows:

Lemma B.1. For $x > 0$, there holds

$$w \circ \arg(x,y) = w \circ \text{crg}(x,y).$$

Proof: Note first that $\text{crg}(x,y) = \tan^{-1}(y/x) \in (-\pi/2, 0)$. If $y \geq 0$, then $\arg(x,y) = \tan^{-1}(y/x)$, and it is clear that $\arg(x,y) = \text{crg}(x,y)$ and consequently $w \circ \arg(x,y) = w \circ \text{crg}(x,y)$ as asserted.

If $y < 0$, then $\arg(x,y) = 3\pi/2 + \tan^{-1}(-x/y) = \pi + \cot^{-1}(x/y)$.

Exercise B.2. Verify the identity

$$3\pi/2 + \tan^{-1}(-x/y) = \pi + \cot^{-1}(x/y)$$

and the other identities justifying the expression (B.2).

In the last expression $x/y < 0$ and $\cot^{-1}(x/y) \in (\pi/2, \pi)$. Thus,

$$\frac{3\pi}{2} < \arg(x,y) = \pi + \cot^{-1}(x/y) < 2\pi$$

and

$$-\frac{\pi}{2} < \arg(x,y) - 2\pi = -\pi + \cot^{-1}(x/y) < 0.$$

Furthermore,

$$\begin{aligned} \tan(\arg(x,y) - 2\pi) &= \tan(-\pi + \cot^{-1}(x/y)) \\ &= \tan\cot^{-1}(x/y) \\ &= y/x. \end{aligned}$$

[2]This is essentially the function approximated with values `Arg[x + I y]` in Mathematica.

We conclude $\arg(x, y) - 2\pi = \mathrm{crg}(x, y)$. Because w is 2π periodic

$$w \circ \arg(x, y) w(\mathrm{crg}(x, y) + 2\pi) = w \circ \mathrm{crg}(x, y)$$

in this case as well, and the lemma is proved. $\square$

As a corollary of the lemma given a point $(x, 0)$ with $x > 0$, there is some $\delta > 0$ for which

$$w \circ \arg(\xi, \eta) = w \circ \mathrm{crg}(\xi, \eta) \qquad \text{for} \qquad (\xi, \eta) \in B_\delta(x, 0).$$

But $\mathrm{crg} \in C^\infty(\mathbb{R}^2 \backslash \{(x, 0) : x < 0\})$. Therefore,

$$w \circ \mathrm{crg}\big|_{B_\delta(x,0)} \in C^k(B_\delta(x, 0))$$

and the radial function $f = w \circ \arg \in C^k(\mathbb{R}^2 \backslash \{(0, 0)\})$ as stipulated by the main assertion.

Exercise B.3. We have given the proof under the special assumption that f is a radial function with domain $\mathbb{R}^2 \backslash \{(0, 0)\}$. Check the details to see what changes are required to treat the more general situation of a radial function $f : U \to \mathbb{R}$ as specified in Definition 9.

Exercise B.4. We have not mentioned above the branches of crg with values given by

$$\mathrm{crg}_j(x, y) = \mathrm{crg}(x, y) + j\pi$$

for $j \in \mathbb{Z}$. Make nice plots of the tangent function $\tan : \mathbb{R} \backslash \{(1/2 + j)\pi : j \in \mathbb{Z}\} \to \mathbb{R}$ and the branches $\tan_j^{-1}$ of the arctangent. Give a piecewise expression for the values of crg_j using only appropriate branches of the arctangent and the arccotangent (and no explicit additive multiples of π.)

The polar coordinates map $\Psi : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ with

$$\Psi(r, \theta) = (r \cos\theta, r \sin\theta)$$

is a local diffeomorphism with convenient "branches" of the inverse defined in terms of the branches of arg and crg. Specifically, one may take for $j \in \mathbb{Z}$ and $(x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\}$

$$\Psi_j^{-1}(x, y) = \left(\sqrt{x^2 + y^2}, \arg_j(x, y)\right) \tag{B.7}$$

or

$$\Psi_j^{-1}(x, y) = \left(\sqrt{x^2 + y^2}, \mathrm{crg}_j(x, y)\right). \tag{B.8}$$

Exercise B.5. Specify the natural domains for the inverse polar coordinates functions with values given in (B.7) and (B.8). Make (nice) illustrations to go along with each of these local inverses.

Appendix C

Subjectivity and subject nativity

We discuss the subjects of calculus, infinite dimensional analogues of elements of calculus on Banach spaces, differential geometry, and linear algebra in a specific context related to understanding and notation of certain aspects of these subjects.

When I learned about the derivative of a real valued function of a real variable for the first time, I came away with the impression that the derivative was the "same kind of object" as the function itself. Specifically given an open interval $(a, b) \subset \mathbb{R}$ and $f : (a, b) \to \mathbb{R}$, I learned the function f is **differentiable at** $x \in (a, b)$ if

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

exists and the value of the limit is called the **derivative of** f **at** x. If the function is differentiable at every point $x \in (a, b)$, so the story goes, then it makes sense to consider the function of one variable $f' : (a, b) \to \mathbb{R}$ wtih values

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

One may then refer to the collection of all such functions as something like $\text{Diff}(a, b)$. There may be various sets of functions to which the functions f and f' belong or do not belong. It is a theorem that $f \in \text{Diff}(a, b)$ implies $f \in C^0(a, b)$ where $C^0(a, b)$ is the collection of all real valued **continuous** functions on (a, b).

A function $f \in C^0(a,b)$ may not be in $\text{Diff}(a,b)$, so one might say f and f' are not the same kind of objects in the sense that they are not necessarily both in some particular function space. But if we introduce the larger collection $\mathbb{R}^{(a,b)}$ of all real valued functions on the interval (a,b), then both f and f' are certainly in this "space," making them in that sense "the same kind of objects."

I should like to note in passing—or perhaps in a sense a bit more centrally important—that this procedure gets repeated, producing under the appropriate circumstances derivatives of higher order f'', f''',...,

$$\frac{d^k f}{dx^x},$$

and so on, with each successive derivative still being a function in $\mathbb{R}^{(a,b)}$ and hence the same kind of object as f. Again, these may not fit nicely in some particular function class within $\mathbb{R}^{(a,b)}$. One may denote for $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$ the k-times differentiable functions in this context by $\text{Diff}^k(a,b)$ and also introduce the convenient and more commonly used collections of **continuously differentiable** functions

$$C^k(a,b) = \left\{ f \in \text{Diff}^k(a,b) : \frac{d^k f}{dx^x} \in C^0(a,b) \right\}.$$

This construction affords an opportunity to assert, within some context at least, that the function f and its derivatives are all precisely the same kinds of objects even with respect to a function space related to differentiability much smaller than $\mathbb{R}^{(a,b)}$. Specifically, there is a nonempty collection of functions

$$C^\infty(a,b) = \cap_{k=1}^\infty C^k(a,b)$$

containing those functions all of whose derivatives exist and are differentiable. Using the theorem mentioned above it follows that

$$C^\infty(a,b) = \cap_{k=1}^\infty \text{Diff}^k(a,b)$$

as well. It is suggested then that many of the functions we can reasonably claim to know something about, polynomial functions for example, are in $C^\infty(a,b)$ or even, in the case of polynomials, in $C^\infty(\mathbb{R})$.

This is a sort of nice framework. It is comforting.

Later on I started to run into situations in which I was told a function and its derivative are not the same kind of objects in the sense I've tried to describe above, and some of these situations suggest perhaps they should not have been considered the same kinds of objects even in the first simplest case. This is a little bit disturbing, and I'll try to describe some of those situations.

One of the first and most obvious of these situations where a function and its derivative are fundamentally different kinds of objects is when the domain of a real valued function f is not an open interval but rather an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$ with $n > 2$. Given an open set $U \subset \mathbb{R}^n$, there are again collections of real valued functions which are **differentiable**. The definition looks rather different:

One says $f : U \to \mathbb{R}$ is **differentiable at $\mathbf{x} \in U$** if there are n limits

$$\lim_{h\to 0} \frac{f(\mathbf{x}+h\mathbf{e}_j) - f(\mathbf{x})}{h}, \qquad j = 1, 2, \ldots, n \tag{C.1}$$

all of which exist and are called *partial derivatives* (at $\mathbf{x}$) **and** these limits satisfy another limiting condition, namely,

$$\lim_{\mathbf{v}\to \mathbf{0}\in\mathbb{R}^n} \frac{f(\mathbf{x}+\mathbf{v}) - f(\mathbf{x}) - L(\mathbf{v})}{|\mathbf{v}|} = 0$$

where $L : \mathbb{R}^n \to \mathbb{R}$ is a linear function with values determined by the partial derivatives:

$$L(\mathbf{v}) = L(v_1, v_2, \ldots, v_n) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\mathbf{x})\ v_j$$

and

$$\frac{\partial f}{\partial x_j}(\mathbf{x}) = \lim_{h\to 0} \frac{f(\mathbf{x}+h\mathbf{e}_j) - f(\mathbf{x})}{h}$$

denotes the value of the limit in (C.1).

An immediate task in this case is the identification of the derivative. Recall that for $f : (a, b) \to \mathbb{R}$, that is to say, $f \in \mathbb{R}^{(a,b)}$ the **derivative at a point** $x \in (a, b)$ is just a number. In the case of $\mathbb{R}^U$ with $U \subset \mathbb{R}^n$ and $n \geq 2$, one does not seem to have a single number but rather n different numbers. One of these numbers does not constitute the derivative at the point $\mathbf{x}$ but only a **partial** derivative at $\mathbf{x}$. It is usual to designate, at least at first, the derivative at $\mathbf{x}$ to be the vector

$$\left(\lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{e}_1)-f(\mathbf{x})}{h}, \lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{e}_1)-f(\mathbf{x})}{h}, \ldots, \right.$$
$$\left. \lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{e}_n)-f(\mathbf{x})}{h}\right) \in \mathbb{R}^n. \qquad \text{(C.2)}$$

Notice I have used the full notation for the limits in (C.2) and not the more compact notation

$$\frac{\partial f}{\partial x_j}(\mathbf{x}) = \lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{e}_j)-f(\mathbf{x})}{h} \qquad \text{(C.3)}$$

for those limits. The full limit expressions without this notation could have been used in the definition of a differentiable function of several variables above, and perhaps that would be more appropriate as the notation of (C.3) suggests the existence of a function

$$\frac{\partial f}{\partial x_j} : U \to \mathbb{R} \qquad \text{(C.4)}$$

which we are not assuming when defining differentiability at a point. We are only assuming

$$\frac{\partial f}{\partial x_j} : \{\mathbf{x}\} \to \mathbb{R}.$$

It was just too cumbersome for me to write out the definition without using the familiar notation for partial derivatives. Notice I even had to use two lines to fit the display (C.2) on the page.

The next immediate step is to make that notation (more) justified by saying $f : U \to \mathbb{R}$ is **differentiable on** U if f is differentiable at each $\mathbf{x} \in U$. Then (C.3) gives the values of a function as in (C.4). This function is of the same kind at $f : U \to \mathbb{R}$, but there is not just one of them to call the derivative; there are n of them called **partial derivatives**. One can say then, and one usually does, that the **full derivative** or the **total derivative** is the vector valued function $Df : U \to \mathbb{R}^n$ with values given in (C.2), that is

$$Df(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x})\right). \qquad \text{(C.5)}$$

This is definitely a new and different kind of object, a different kind of object from $f : U \to \mathbb{R}$.

One the face of it, this derivative Df is a vector valued function $Df : U \to \mathbb{R}^n$. It should be noted, however, that thinking of the derivative this way depends on having a basis (with finitely many vectors) for the space $\mathbb{R}^n$ containing the open set U. There are various contexts where this condition of having a basis fails to hold.

One such situation is when one considers U as an open subset of a Banach space X. There is also a notion of differentiability in this case, but the linear mapping in the definition does not necessarily have a nice form in terms of something that can be called a derivative. One simply says a function $f : U \to \mathbb{R}$ in this case is **differentiable at** $x \in U$ if there exists some continuous linear function $L : X \to \mathbb{R}$ for which

$$\lim_{v \to \mathbf{0} \in X} \frac{f(x+v) - f(x) - L(v)}{\|v\|} = 0. \tag{C.6}$$

It is customary to call the linear map $L : X \to \mathbb{R}$ the "derivative at x" in this case or the "Fréchet derivative." Since this kind of linear map has a distinct and serviceable name back in the context of $f : U \to \mathbb{R}$ when $U \subset \mathbb{R}^n$, this seems like a rather objectionable choice for the terminology.

Returning for a moment to $f : U \to \mathbb{R}$ with U and open set in $\mathbb{R}^n$, the mapping L given in the definition of differentiability is also called the **differential** or differential map, so that $L = df_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}$. Again, this is the differential and not the derivative.

It seems like some kind of consistency can be easily maintained if one calls the linear map $L : X \to \mathbb{R}$ when X is a Banach space, $f : U \to \mathbb{R}$ with U an open subset of X and (C.6) holds, the **differential** df_x of f at x. One could also call this linear map the **Fréchet differential** which is sometimes done. In this case there may be no derivative of the function per se but only a differential.

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satisfying
$\text{Diff}^k(U)$, which are differentiable.

Appendix D

Hopf patch

The situation with regard to the definition of a C^k regular embedded surface as distinguished from the definition of local surface given by Hopf may be clarified by the following definition.

Definition 10. (Hopf patch, a.k.a. regular Osserman patch) Given U an open subset of $\mathbb{R}^2$ and $k \in \mathbb{N}$, a function $X \in C^k(U \to \mathbb{R}^3)$ is said to be a **Hopf patch** if $dX_{\mathbf{u}} : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one for each $\mathbf{u} \in U$.

Theorem 7. (Gray's theorem) Given a C^k Hopf patch $X : U \to \mathbb{R}^3$ and given $\mathbf{u} \in U$, there is some $\delta > 0$ such that $B_\delta(\mathbf{u}) \subset U$ and for any open set $\tilde{U} \subset B_\delta(\mathbf{u})$, the image $X(\tilde{U})$ is a regular embedded surface in the sense of Definition 2 (and hence also in the sense of Definition 1 once it is established that Definition 1 and Definition 2 are equivalent). Also, one may assume, or arrange to assume,

$$\tilde{X} = X\big|_{\tilde{U}} : \tilde{U} \to \mathbb{R}^3$$

is an admissible global parameterization for the surface $X(\tilde{U})$. In particular, since $X(B_\delta(\mathbf{u}))$ is one such C^k embedded surface, every function $\tilde{X}$ may be assumed to be a local parameterization for the surface $X(B_\delta(\mathbf{u}))$.

Proof: Gray's proof is essentially correct though the following construction may bear the patina of being a little more general.

We have a well-defined (non-vanishing) C^{k-1} function $N : U \to \mathbb{S}^2 \subset \mathbb{R}^3$ given by

$$N = \frac{X_{u_1} \times X_{u_2}}{|X_{u_1} \times X_{u_2}|}.$$

Denote by N_p the value $X(\mathbf{u})$. Note that N_p is a fixed unit vector in $\mathbb{R}^3$. There are many unit vectors $E_1, E_2 \in \mathbb{S}^2$ for which $\{E_1, E_2, N_p\}$ is an orthonormal basis for $\mathbb{R}^3$ with $E_1 \times E_2 = N_p$. For example, if $N_p \neq \mathbf{e}_3$, take $E_2 = N_p \times \mathbf{e}_3 / |N_p \times \mathbf{e}_3|$ and $E_1 = E_2 \times N_p$. Let $\{E_1, E_2\}$ be one such an ordered orthonormal basis.

Consider $\psi : U \to \mathbb{R}^2$ by $\psi(x) = ((X - p) \cdot E_1, (X - p) \cdot E_2)$. Then

$$D\psi = \begin{pmatrix} X_{u_1} \cdot E_1 & X_{u_2} \cdot E_1 \\ X_{u_1} \cdot E_2 & X_{u_2} \cdot E_2 \end{pmatrix}$$

and

$$D\psi(\mathbf{u}) = \begin{pmatrix} X_{u_1}(\mathbf{u}) \cdot E_1 & X_{u_2}(\mathbf{u}) \cdot E_1 \\ X_{u_1}(\mathbf{u}) \cdot E_2 & X_{u_2}(\mathbf{u}) \cdot E_2 \end{pmatrix}.$$

The vectors $X_{u_1}(\mathbf{u})$ and $X_{u_2}(\mathbf{u})$ satisfy

$$X_{u_1}(\mathbf{u}) \cdot N_p = 0 \qquad \text{and} \qquad X_{u_2}(\mathbf{u}) \cdot N_p = 0.$$

Thus the j-th column in $D\psi(\mathbf{u})$ contains the first two components of the expansion of X_{u_j} in the basis $\{E_1, E_2, N_p\}$ for $j = 1, 2$ with the last component(s) being zero. This means in particular

$$\begin{aligned}
\mathbf{0} &\neq X_{u_1}(\mathbf{u}) \times X_{u_2}(\mathbf{u}) \\
&= [(X_{u_1}(\mathbf{u}) \cdot E_1)E_1 + (X_{u_1}(\mathbf{u}) \cdot E_2)E_2] \\
&\qquad\qquad \times [(X_{u_2}(\mathbf{u}) \cdot E_1)E_1 + (X_{u_2}(\mathbf{u}) \cdot E_2)E_2] \\
&= (X_{u_1}(\mathbf{u}) \cdot E_1)(X_{u_2}(\mathbf{u}) \cdot E_2)E_1 \times E_2 + (X_{u_1}(\mathbf{u}) \cdot E_2)(X_{u_2}(\mathbf{u}) \cdot E_1)E_2 \times E_1 \\
&= [(X_{u_1}(\mathbf{u}) \cdot E_1)(X_{u_2}(\mathbf{u}) \cdot E_2) - (X_{u_1}(\mathbf{u}) \cdot E_2)(X_{u_2}(\mathbf{u}) \cdot E_1)]N_p \\
&= \det D\psi(\mathbf{u})\ N_p.
\end{aligned}$$

Since $N_p \neq \mathbf{0}$ it follows that $\det D\psi(\mathbf{u}) \neq 0$, and by the inverse function theorem, there is some $\epsilon > 0$ such that

$$\psi_0 = \psi\Big|_{B_\epsilon(\mathbf{u})} : B_\epsilon(\mathbf{u}) \to Y = \psi(B_\epsilon(\mathbf{u}))$$

is a C^k diffeomorphism. In particular, Y is also an open set in $\mathbb{R}^2$. We denote by $\sigma : Y \to B_\delta(\mathbf{u})$ the inverse map.

Notice that $\psi(\mathbf{u}) = \mathbf{0} \in Y$ and consider the open cylinder

$$V = \left\{ p + y_1 E_1 + y_2 E_2 + y_3 N_p : \mathbf{y} = (y_1, y_2) \in Y \text{ and } y_3 \in \mathbb{R} \right\}.$$

We wish to show first

$$X\big|_{B_\delta(\mathbf{u})} : B_\delta(\mathbf{u}) \to \mathcal{S} = X(B_\delta(\mathbf{u})) = V \cap \mathcal{S}$$

is a homeomorphism. This is enough to establish $\mathcal{S} = X(B_\delta(\mathbf{u}))$ is a C^k em-

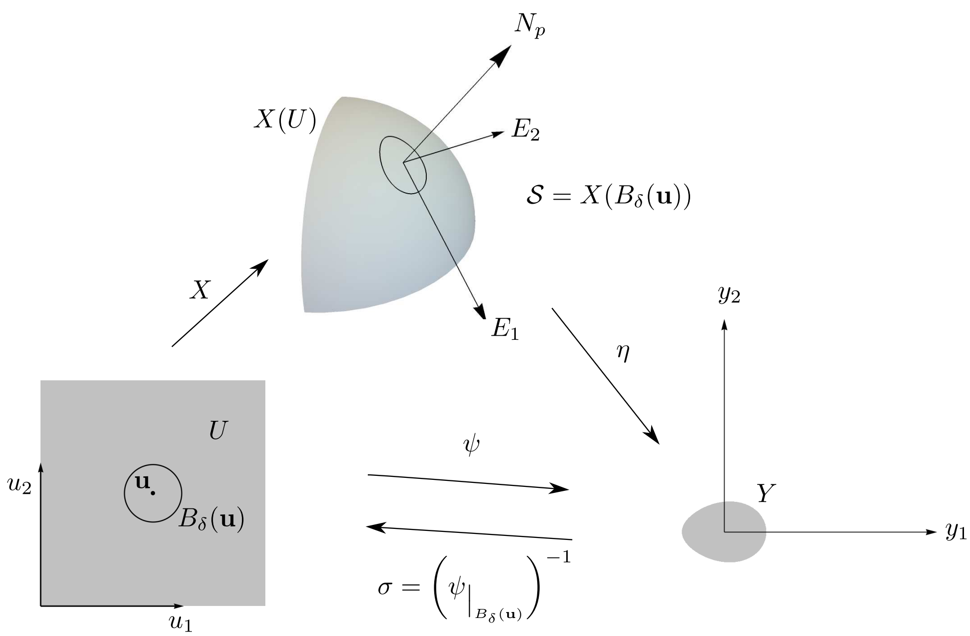

Figure D.1: A Hopf mapping X. The image of the mapping $X(B_\delta(\mathbf{u}))$ of a small enough ball is a surface.

bedded surface with a single permissive parameterization X in the language of Definition 2.

First observe that $X(B_\delta(\mathbf{u})) \subset V \cap X(B_\delta(\mathbf{u}))$ so that

$$X(B_\delta(\mathbf{u})) = V \cap X(B_\delta(\mathbf{u}))$$

as stated above. In fact, for each $\mathbf{w} \in B_\delta(\mathbf{u})$ we know

$$\psi(\mathbf{w}) = ((X(\mathbf{w}) - p) \cdot E_1, (X(\mathbf{w}) - p) \cdot E_2) \in Y.$$

Writing $\mathbf{y} = (y_1, y_2) = \psi(\mathbf{w}) \in Y$, we observe

$$\begin{aligned} X(\mathbf{w}) &= p + (X(\mathbf{w}) - p) \\ &= p + [(X(\mathbf{w}) - p) \cdot E_1]\ E_1 + [(X(\mathbf{w}) - p) \cdot E_2]\ E_2 \\ &\qquad\qquad + [(X(\mathbf{w}) - p) \cdot N_p]\ N_p \\ &= p + y_1\ E_1 + y_2\ E_2 + [(X(\mathbf{w}) - p) \cdot N_p]\ N_p. \end{aligned}$$

Since $\mathbf{y} = (y_1, y_2) \in Y$ and $y_3 = (X(\mathbf{w}) - p) \cdot N_p \in \mathbb{R}$, this shows $X(\mathbf{w}) \in V$, and consequently $X(B_\delta(\mathbf{u})) \subset V \cap X(B_\delta(\mathbf{u}))$ as claimed.

Next we verify

$$X_0 = X\Big|_{B_\delta(\mathbf{u})} : B_\delta(\mathbf{u}) \to \mathcal{S}$$

is one-to-one. If $\mathbf{w}, \tilde{\mathbf{w}} \in B_\delta(\mathbf{u})$ with $X(\mathbf{w}) = X(\tilde{\mathbf{w}})$, then

$$\begin{aligned} \psi(\mathbf{w}) &= ((X(\mathbf{w}) - p) \cdot E_1, (X(\mathbf{w}) - p) \cdot E_2) \\ &= ((X(\tilde{\mathbf{w}}) - p) \cdot E_1, (X(\tilde{\mathbf{w}}) - p) \cdot E_2) \\ &= \psi(\tilde{\mathbf{w}}). \end{aligned}$$

Applying the inverse σ we conclude $\tilde{\mathbf{w}} = \mathbf{w}$ and X_0 is one-to-one.

Since $\mathcal{S} = X(B_\delta(\mathbf{u})) = X(B_\delta(\mathbf{u})) \cap V$, it is obvious that $X_0 : B_\delta(\mathbf{u}) \to \mathcal{S} \cap V$ is surjective. Thus, we have an inverse $\xi_0 : \mathcal{S} \to B_\delta(\mathbf{u})$ and it remains to show $\xi = \xi_0$ is continuous.

Consider the function $\eta : V \to \mathbb{R}^2$ with values

$$\eta(\mathbf{x}) = ((\mathbf{x} - p) \cdot E_1, (\mathbf{x} - p) \cdot E_2).$$

It is clear $\eta \in C^0(V \to \mathbb{R}^2)$. Furthermore, if $\mathbf{x} \in V$, then

$$\mathbf{x} = p + y_1 E_1 + y_2 E_2 + y_3 E_3$$

for some $\mathbf{y} = (y_1, y_2) \in Y$ and $y_3 \in \mathbb{R}$. By taking dot products, we find

$$(\mathbf{x} - p) \cdot E_1 = y_1 \qquad \text{and} \qquad (\mathbf{x} - p) \cdot E_2 = y_2.$$

This means $\eta(\mathbf{x}) = (y_1, y_2) \in Y$ and $\eta : V \to Y$. We claim

$$\xi_0 = \sigma \circ \eta\Big|_{\mathcal{S}}.$$

Since the composition $\sigma \circ \eta \in C^0(V \to B_\delta(\mathbf{u}))$ this is the final step in showing $X_0 : B_\delta(\mathbf{u}) \to \mathcal{S}$ is a homeomorphism. Denote by η_0 the restriction

$$\eta_0 = \eta\Big|_{\mathcal{S}} : \mathcal{S} \to Y.$$

Notice it is clear $\sigma \circ \eta(q) \in B_\delta(\mathbf{u})$ for every $q \in \mathcal{S}$. In particular $X \circ \sigma \circ \eta(q)$ is well-defined. On the other hand,

$$q = p + [(q-p) \cdot E_1]\ E_1 + [(q-p) \cdot E_2]\ E_2 + [(q-p) \cdot N_p]\ N_p. \qquad \text{(D.1)}$$

It follows that $\eta(q) = ((q-p) \cdot E_1, (q-p) \cdot E_2) = \mathbf{y} \in Y$, and $\mathbf{w} = \sigma \circ \eta(q) \in B_\delta(\mathbf{u})$ is the unique point $\mathbf{w} \in B_\delta(\mathbf{u})$ for which

$$\psi(\mathbf{w}) = \eta(q) = ((q-p) \cdot E_1, (q-p) \cdot E_2).$$

On the other hand, $\xi_0(q) \in B_\delta(\mathbf{u})$ and $q = X \circ \xi_0(q)$ so that

$$\begin{aligned} \psi(\xi_0(q)) &= ((X \circ \xi_0(q) - p) \cdot E_1, (X \circ \xi_0(q) - p) \cdot E_2) \\ &= ((q-p) \cdot E_1, (q-p) \cdot E_2) \end{aligned}$$

as well. This means $\xi_0(q) = \mathbf{w} = \sigma \circ \eta(q)$ and $\xi_0 = \sigma \circ \eta_0 \in C^0(\mathcal{S} \to B_\delta(\mathbf{u}))$.

starting with (D.1) the last argument can be perhaps simplified a little bit as follows: Note first that since σ is the inverse of ψ_0 one has

$$\begin{aligned} \psi \circ \sigma \circ \eta(q) &= \eta(q) \\ &= ((q-p) \cdot E_1, (q-p) \cdot E_2). \end{aligned}$$

On the other hand,

$$\begin{aligned} \psi(\xi_0(q)) &= ((X \circ \xi_0(q) - p) \cdot E_1, (X \circ \xi_0(q) - p) \cdot E_2) \\ &= ((q-p) \cdot E_1, (q-p) \cdot E_2). \end{aligned}$$

Thus, for each $q \in \mathcal{S}$ there holds

$$\psi_0 \circ \sigma \circ \eta(q) = \psi_0 \circ \xi_0(q).$$

Applying the inverse σ to both sides, $\xi_0(q) = \sigma \circ \eta(q)$ so

$$\xi_0 = \sigma \circ \eta_0 \in C^0(\mathcal{S} \to B_\delta(\mathbf{u})).$$

Finally, we consider an arbitrary open set $\tilde{U} \subset B_\delta(\mathbf{u})$ to which most of the constructions above apply. If one wishes to go through the full details, the situation is simplifed to a certain extent; repeated use of the inverse function theorem is not necessary as the C^k diffeomorphism

$$\psi\big|_{B_\delta(\mathbf{u})} : B_\delta(\mathbf{u}) \to Y = \psi(B_\delta(\mathbf{u}))$$

may simply be replaced with

$$\tilde{\psi} = \psi\big|_{\tilde{U}} : \tilde{U} \to \tilde{Y} = \psi(\tilde{U}).$$

In this case however we can use the fact that $X_0 : B_\delta(\mathbf{u}) \to \mathcal{S}$ is an open map and take an open set $\tilde{V}$ for which $X(\tilde{U}) = X(\tilde{U}) \cap \tilde{V}$. Alternatively, we can make the specific choice

$$\tilde{V} = \left\{ p + y_1 E_1 + y_2 E_2 + y_3 N_p : \mathbf{y} = (y_1, y_2) \in \tilde{Y} \text{ and } y_3 \in \mathbb{R} \right\}.$$

Either way $X(\tilde{U})$ is an open set in $\mathcal{S} = X(B_\delta(\mathbf{u}))$ and

$$\tilde{\xi} = \xi_0\big|_{X(\tilde{U})} : X(\tilde{U}) \to \tilde{U}$$

provides a continuous inverse for

$$\tilde{X} = X\big|_{\tilde{U}} : \tilde{U} \to X(\tilde{U}).$$

Thus, $\tilde{X}$ is a permissive local parameterization for $\mathcal{S} = X(B_\delta(\mathbf{u}))$ and of course a global parameterization of the surface $X(\tilde{U})$. $\square$

Appendix E

Large initial covering collections

NOTE: I wrote the appendix below with the intent of constructing a strict initial covering collection of a surface which could not be indexed by the points in the surface. The idea was that given a single (strict initial) parameterization $X = X_p : \mathbb{R} \to B_{\epsilon_p}(p) \cap \mathcal{S}$ from the definition, one could find many more distinct (strict initial) parameterizations at p essentially by using diffeomrphisms $\psi : B_{\epsilon_p}(p) \to B_r(p)$ for various disks $B_r(p)$ and maps ψ. While there are "many" such maps, collections of continuous maps (like diffeomorphisms) tend to not have so so many elements. The cardinality is usually the same as the cardinality of $\mathbb{R}$, unlike the cardinality of collections of arbitrary functions on open subsets of $\mathbb{R}^n$ which tend to have strictly higher cardinality. As a result, I think I'm going to have to back pedal on my assertion that one can't expect to index an initial covering collection of a surface using the points in the surface. It may even be the case that the maximal covering collection can always be indexed by the points in the surface simply because this collection has the same cardinality as the surface. I'm not quite sure about that, and I'm also not quite sure it's worth reading what is below. But if you want to have a look at my apparently unsuccessful efforts, here is how the appendix was drafted:

I do not intend to suggest by this remark that a given C^k embedded surface cannot have an initial covering collection indexed by the points p in the surface.

Exercise E.1. Verify that $X = X_{\mathbf{e}_3} : \mathbb{R}^2 \to \mathbb{S}^2 \backslash \{-\mathbf{e}_3\}$ by

$$X(u_1, u_2) = \frac{(2u_1, 2u_2, 1 - u_1^2 - u_2^2)}{1 + u_1^2 + u_2^2}$$

is a strict initial parameterization for $\mathbb{S}^2$ with $X(\mathbf{0}) = \mathbf{e}_3 \in \mathbb{S}^2$.

More generally, let $p \in \mathbb{S}^2$ be fixed and also fix vectors $E_1, E_2 \in \mathbb{S}^2$ so that $\{E_1, E_2, p\}$ is an orthonormal basis with $E_1 \times E_2 = p$. Verify $X = X_p : \mathbb{R}^2 \to \mathbb{S}^2 \backslash \{-p\}$ by

$$X(u_1, u_2) = \frac{(2u_1 E_1, 2u_2 E_2, (1 - u_1^2 - u_2^2) p)}{1 + u_1^2 + u_2^2}$$

is a strict initial parameterization for $\mathbb{S}^2$ with $X(\mathbf{0}) = p \in \mathbb{S}^2$.

Conclude $\{X_p\}_{p \in \mathbb{S}^2}$ is an initial covering collection for $\mathbb{S}^2$.

Exercise E.2. Give a permissive covering collection (Definition 2) for $\mathbb{S}^2$ using fewer parameterizations than there are points in $\mathbb{S}^2$.

$\mathcal{S} = \{(x_1, x_2, 0) : (x_1, x_2) \in \mathbb{R}^2\}$ is also an embedded surface. I want to construct an initial covering collection for this plane $\mathcal{S}$ with more parameterizations in it than there are points in the plane. There are "trivial" ways to do this involving indexing, but I want to do something a little less trivial.

Let $p = (p_1, p_2, 0) \in \mathcal{S}$ be fixed. I can start with $X_0(u_1, u_2) = p + (u_1, u_2, 0)$. Note that $X(\mathbf{0}) = p$ and $X_{u_j} = \mathbf{e}_j$ for $j = 1, 2$. The inverse of $X_0 : \mathbb{R}^2 \to \mathcal{S}$ is given by

$$\xi(x_1, x_2, 0) = (x_1, x_2, 0) - p.$$

This fails to be a strict initial parameterization according to Definition 1 because the image $X_0(\mathbb{R}^2)$ is not of the form $B_\epsilon(p) \cap \mathcal{S}$ for any $\epsilon > 0$ and $B_\epsilon(p) \subset \mathbb{R}^3$. I guess this can be fixed by using the inverse of $\psi : [0, 1)\mathbf{0}) \cap [0, \infty)$ by $\psi(x) = x/(1 - x^2)$. That is,

$$\psi^{-1}(y) = \begin{cases} \dfrac{-1 + \sqrt{1 + 5y^2}}{2y}, & y \neq 0 \\ \\ 0, & y = 0. \end{cases}$$

For example, the function $\Psi : B_1(\mathbf{0}) \to \mathbb{R}^2$ where $B_1(\mathbf{0}) \subset \mathbb{R}^2$ by $\Psi(\mathbf{u}) = \psi(|\mathbf{u}|)\mathbf{u}/|\mathbf{u}|$ is a C^∞ diffeomorphism with $\Psi^{-1}(\mathbf{y}) = \psi^{-1}(|\mathbf{y}|)\mathbf{y}/|\mathbf{y}|$.

Similarly, I'll take $X : \mathbb{R}^2 \to \mathcal{S}$ with

$$X(\mathbf{u}) = p + \psi^{-1}(|\mathbf{u}|)(u_1, u_2, 0)/|\mathbf{u}|.$$

This gives a strict initial parameterization with $X(\mathbb{R}^2) = \mathcal{S} \cap B_1(p)$. Let's call this initial parameterization X_{pp}.

For each $q = (q_1, q_2, 0) \in \mathcal{S}\backslash\{p\}$, there is a unique angle $\theta = \theta(q) \in [0, 2\pi)$ for which

$$\cos\theta = \frac{(q-p)\cdot \mathbf{e}_1}{|q-p|} \qquad \text{and} \qquad \sin\theta = \frac{(q-p)\cdot \mathbf{e}_2}{|q-p|}.$$

Thus, for each $q = (q_1, q_2, 0) \in \mathcal{S}\backslash\{p\}$ we may define $X = X_{pq} : \mathbb{R}^2 \to B_{|q-p|}(p) \subset \mathbb{R}^3$ by

$$X(\mathbf{u}) = p + |q-p|\psi^{-1}(|\mathbf{u}|)(u_1\cos\theta - u_2\sin\theta, u_1\sin\theta + u_2\cos\theta, 0)/|\mathbf{u}|.$$

I think that gives a strict initial parameterization, and X_{pq} is clearly different from $X_{\tilde{p}\tilde{q}}$ as long as $(p, q) \neq (\tilde{p}, \tilde{q})$. Thus,

$$\{X_{pq}\}_{(p,q)\in\mathbb{R}^2}$$

is an initial covering collection of strict initial parameterizations. Technically, however, I suppose one could relatively easily re-index this collection using the points $p \in \mathcal{S}$.

Exercise E.3. Show that any strict initial covering collection for a surface $\mathcal{S}$ has at least one distinct parameterization for each point $p \in \mathcal{S}$, but the sphere $\mathcal{S}^2$ can be "naturally" covered using 2, 3, 4, 5, or 6 permissive initial covering maps.

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