

# COUNTEREXAMPLES FOR LOCAL ISOMETRIC EMBEDDING

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## 1. INTRODUCTION

In this paper, we construct metrics on 2-manifold which cannot be even locally isometrically embedded in the Euclidean space  $\mathbb{R}^3$ . By isometric embedding of  $(M^2, g)$  with  $g = \sum_{i,j=1}^2 g_{ij} dx_i dx_j$  in  $\mathbb{R}^3$ , we mean there exists a surface in  $\mathbb{R}^3$  with the induced metric equaling  $g$ , namely, the three coordinate functions  $(X(x_1, x_2), Y(x_1, x_2), Z(x_1, x_2))$  defined on  $M^2$  satisfy

$$dX^2 + dY^2 + dZ^2 = \sum_{i,j=1}^2 g_{ij} dx_i dx_j.$$

To be precise, we state the results in the following

**Theorem 1.1.** *There exists a smooth metric  $g$  in  $B_1 \subset \mathbb{R}^2$  with Gaussian curvature  $K_g \leq 0$  such that there is no  $C^3$  isometric embedding of  $(B_r(0), g)$  in  $\mathbb{R}^3$  for any  $r > 0$ .*

**Theorem 1.2.** *There exists a smooth metric  $g$  in  $B_1 \subset \mathbb{R}^2$  with Gaussian curvature  $K_g(0) = 0$  and  $K_g(x) < 0$  for  $x \neq 0$  such that there is no  $C^{3,\alpha}$  isometric embedding of  $(B_r(0), g)$  in  $\mathbb{R}^3$  for any  $r > 0$  and  $\alpha > 0$ .*

Pogorelov [P2] constructed a simple  $C^{2,1}$  metric  $g$  in  $B_1 \subset \mathbb{R}^2$  with sign-changing Gaussian curvature such that  $(B_r, g)$  cannot be realized as a  $C^2$  surface in  $\mathbb{R}^3$  for any  $r > 0$ . Recently the first author [N] gave a  $C^\infty$  metric  $g$  on  $B_1$  with no smooth isometric embedding of  $(B_r, g)$  in  $\mathbb{R}^3$  for any  $r > 0$ . The sign of the Gaussian curvature  $K_g$  also changes.

On the positive side, when the sign of  $K_g$  for any smooth metric  $g$  does not change, the local smooth isometric embedding was settled by Pogorelov [P1], Nirenberg [Ni], and Hartman and Winter [HW2]. When  $K_g \geq 0$  for the  $C^k$  metric with  $k \geq 10$ , there is a  $C^{k-6}$  isometric embedding of  $(B_{r_k}, g)$  in  $\mathbb{R}^3$ , this was done by Lin [L1]. When  $K_g$  changes sign cleanly, namely,  $K_g(0) = 0, \nabla g(0) \neq 0$  for a  $C^k$  metric  $g$ , Lin [L2] showed that there exists a  $C^{k-3}$  isometric embedding in  $\mathbb{R}^3$  for  $(B_{r_k}, g)$  with  $k \geq 6$ . When  $K_g \leq 0$  and  $\nabla^2 K_g(0) \neq 0$  for the smooth metric  $g$ , there is a local smooth isometric

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embedding of  $g$  in  $\mathbb{R}^3$ , see Iwasaki [I]. When  $K_g = -x_1^{2m} \tilde{K}(x)$  with  $\tilde{K}(0) > 0$  for the smooth metric  $g$ , the same local isometric embedding also holds, see Hong [H]. Recently, Han, Hong, and Lin [HHL] showed that the local isometric embedding exists under the assumption  $K_g \leq 0$  with a certain non-degeneracy of the gradient of  $K_g$ , or  $K_g \leq 0$  with finite order vanishing.

If one allows higher dimensional ambient space, say  $\mathbb{R}^4$ , Poznyak [Po1] proved that any smooth metric  $g$  on  $M^2$  can be locally smoothly isometrically embedded in  $\mathbb{R}^4$ . In fact, any  $C^k$  metric on  $n$ -manifold  $M^n$  has a  $C^k$  global isometric embedding in  $\mathbb{R}^{N_n}$  with  $N_n$  large for  $3 \leq k \leq \infty$ . This is the work by Nash [Na2].

If we start with an analytic metric  $g$  on  $M^n$ , one always has a local analytic isometric embedding of  $(M^n, g)$  in  $\mathbb{R}^{n(n+1)/2}$ . This was proved by Janet [J], Cartan [C] very earlier on, and initiated by Schlaefli in 1873!

Lastly, any  $C^0$  metric  $g$  on a compact  $n$ -manifold  $M^n$  which can be differentially embedded in  $\mathbb{R}^{n+1}$  has a  $C^1$  isometric embedding in  $\mathbb{R}^{n+1}$ , see Nash [Na1] and Kuiper [K].

For general description and further results on isometric embedding problem, we refer to [GR], [P2] and [Y].

The heuristic idea of the construction is to arrange the metric  $g$  in  $B_1$  so that the second fundamental form of any isometric embedded surface in  $\mathbb{R}^3$ ,  $\text{II} \circ i$  vanishes at one point, where  $i : (B_1, g) \rightarrow \mathbb{R}^3$  is the isometric embedding which is supposed to exist. Further we force  $\text{II} \circ i$  to vanish along the boundary of a small domain  $\Omega$  near the center of  $B_1$ , where the Gaussian curvature  $K_g < 0$  (in  $\Omega$ ). By the maximal principle, one cannot have a saddle surface with vanishing second fundamental form along the boundary. So  $(\Omega, g)$  cannot be realized in  $\mathbb{R}^3$ . We repeat the construction near the center of  $B_1$  at every scale so that  $(B_1, g)$  is not isometrically embeddable in  $\mathbb{R}^3$  near the center.

The way to force  $\text{II} \circ i$  to vanish at one point, say  $o$ , is the following. We modify the flat metric  $g_0 = dx^2$  in  $\mathbb{R}^2$  only over certain region  $\Lambda$  slightly away from the center  $o$  to a new one  $g$  so that, for a segment  $A_1 A_2$  with  $A_1, A_2 \in \partial\Lambda$ , the length of  $A_1 A_2$  under  $g$  is shorter than the one of the geodesic  $A_1 A_2$  under the flat  $g_0$ , and  $K_g \leq 0$  in a subregion  $\Lambda_s$  containing  $A_1 A_2$ . Because of  $\det \text{II}(i(0)) = 0$ , we only need to deal with the other principle curvature. Suppose the second one  $\kappa_2 \neq 0$ , say  $\kappa_2 < 0$ . We show that there is a flat concave cylinder  $\Sigma$  near  $i(B_1)$ , which is isometric to  $(B_1, g_0)$  provided the embedding  $i$  is  $C^3$  (This assertion for  $C^2$  embedding case remains unclear to us). Now  $i(A_1 A_2)$  supported on the saddle surface  $i(\Lambda_s)$  can only stay above the concave cylinder  $\Sigma$ . Then the length of  $i(A_1 A_2)$  is longer than the one of the projection of  $i(A_1 A_2)$  down to the flat  $\Sigma$ , call it  $P \circ i(A_1 A_2)$ . We know the length of  $P \circ i(A_1 A_2)$  under  $g_0$  is equal to or longer than that of the geodesic  $A_1 A_2$  under  $g_0$ . But we start from  $A_1 A_2$  with shorter length under  $g$  than under  $g_0$ . This contradiction shows that  $\text{II} \circ i(0)$  vanishes.

Inevitably,  $K_g$  is positive somewhere in  $\Lambda$  if  $\Lambda$  is surrounded by flat region with metric  $dx^2$ . We add “tails” extending to the boundary  $\partial B_1$  for the

modifying regions  $\Lambda$ , modify the metric on the tails, then we have the  $g$  with  $K_g \leq 0$  in  $B_1$ . It turns out that we cannot work with a segment in the construction, we go with a minimal tree connecting three points on  $\partial\Lambda$  for each  $\Lambda$ , see section 2 for details.

Now that we have a non-isometrically embeddable metric (with nonpositive Gaussian curvature), the nearby metrics are almost non-isometrically embeddable. Based on this observation, we construct a non-isometrically embeddable metric with negative Gaussian curvature except for one point in section 3.

## 2. METRIC WITH NONPOSITIVE CURVATURE

Recall any three segments in  $\mathbb{R}^2$  with equal angles  $\frac{2}{3}\pi$  at the common vertex form a minimal tree  $T$ , namely, the length of  $T$  is less than that of any arcs connecting the other three vertices.

**Lemma 2.1.** *Let  $u = -\operatorname{Im} e^{\log^2 z} = -e^{\log^2 r - \theta^2} \sin(2\theta \log r)$ ,  $0 < \theta < 2\pi$ . Then there exists a large integer  $K$  such that*

$$\int_T u ds < 0,$$

where the minimal tree  $T = AA_1 \cup AA_2 \cup AA_3$  with  $A = (-e^{-K}, 0)$ ,  $A_2 = (-1, 0)$ ,  $A_1, A_2 \in \partial B_1$ ,  $\angle A_1AA_2 = \angle A_2AA_3 = \frac{2}{3}\pi$ . Moreover,  $u_r < 0$  for  $r = 1$ .

*Proof.* Set  $\Omega_u = B_1 \cap \text{Sector} A_1AA_2$ ,  $\Omega_l = B_1 \cap \text{Sector} A_2AA_3$ ,  $\widehat{A_1A_2} = \partial\Omega_u \cap \partial B_1$ ,  $\widehat{A_2A_3} = \partial\Omega_l \cap \partial B_1$ . Let the angle from  $A_1A$  to  $x$  be  $\varphi$ , or  $\varphi(x) = \angle A_1Ax$ , then  $0 \leq \varphi(x) \leq \frac{4}{3}\pi$  for  $x \in \Omega_u \cup \Omega_l$ .

We apply Green formula to harmonic functions  $u$  and  $\varphi$  in  $\Omega_u$  and  $\Omega_l$ ,

$$\begin{aligned} \int_{\partial\Omega_u} u \varphi_\gamma ds &= \int_{\partial\Omega_u} \varphi u_\gamma ds \\ \int_{\partial\Omega_l} u \left( \varphi - \frac{4}{3}\pi \right)_\gamma ds &= \int_{\partial\Omega_l} \left( \varphi - \frac{4}{3}\pi \right) u_\gamma ds, \end{aligned}$$

where  $\gamma$  is the outward unit normal of the integral domain. We then have

$$\begin{aligned} \int_{AA_1} -uds + \int_{AA_2} uds &= \int_{\widehat{A_1A_2}} \varphi u_r ds + \int_{AA_2} \frac{2}{3}\pi u_\theta ds \\ \int_{AA_2} -uds + \int_{AA_3} uds &= \int_{\widehat{A_2A_3}} \left( \varphi - \frac{4}{3}\pi \right) u_r ds + \int_{AA_2} \frac{2}{3}\pi u_\theta ds. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{AA_1 \cup AA_3} uds &= 2 \int_{AA_2} uds + \int_{\widehat{A_1A_2}} -\varphi u_r ds + \int_{\widehat{A_2A_3}} \left( \varphi - \frac{4}{3}\pi \right) u_r ds \\ &= 2 \int_{AA_2} uds + \int_{\widehat{A_1A_2}} \varphi e^{-\theta^2} 2\theta ds + \int_{\widehat{A_2A_3}} \left( \frac{4}{3}\pi - \varphi \right) e^{-\theta^2} 2\theta ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{AA_2} u ds &= \int_{e^{-K}}^{e^0} -e^{(\log^2 r - \pi^2)} \sin(2\pi \log r) dr \\ &= \frac{1}{2\pi e^{\pi^2}} \int_{-2\pi K}^0 -e^{\left(\frac{t^2}{4\pi^2} + \frac{t}{2\pi}\right)} \sin t dt. \end{aligned}$$

We choose large enough integer  $K$  so that  $\int_{AA_2} u ds < 0$  and

$$2 \int_{AA_2} u ds + \int_{\widehat{A_1 A_2}} \varphi e^{-\theta^2} 2\theta ds + \int_{\widehat{A_2 A_3}} \left(\frac{4}{3}\pi - \varphi\right) e^{-\theta^2} 2\theta ds < 0.$$

Therefore

$$\int_T u ds < 0.$$

□

**Remark.** By applying Green formula to the above harmonic function  $u$  and linear functions, one sees that  $\int_{\Gamma} u ds > 0$  for any segment  $\Gamma \subset \Omega_u \cup \Omega_l$ , connecting two boundary points on  $\partial B_1$ .

**Lemma 2.2.** *There exists a function  $v \in C_0^\infty(B_{1.1})$  satisfying*

$$\begin{aligned} v &= 0 & \text{in } \{(x_1, x_2) \mid x_1 < 0.9\} \setminus B_1 \\ \Delta v &\geq 0 & \text{in } B_1 \\ \int_T v ds &< 0 \end{aligned}$$

where the minimal tree  $T = CC_1 \cup CA_2 \cup CC_3$  with  $A_2 = (-1, 0)$ ,  $C = (-\frac{1}{10}e^{-K} - 0.8, 0)$ ,  $C_1, C_3 \in \partial B_1$  and  $\angle C_1 CA_2 = \angle A_2 CC_3 = \frac{2}{3}\pi$ . Moreover  $T \subset \{(x_1, x_2) \mid x_1 < -0.1\}$ .

*Proof.* Set  $D = (-e^{-2K}, 0)$ ,  $D_1, D_2 \in \partial B_1$  with  $\angle D_1 DA_2 = \angle A_2 DD_3 = \frac{2}{3}\pi$ , and  $D_4 = (20, x_2(D_3))$ ,  $D_5 = (20, x_2(D_1))$ . Set  $\Omega_p = \text{Pentagon } D_1 DD_3 D_4 D_5$ . Let  $w$  satisfy

$$\begin{aligned} \Delta w &= 0 & \text{in } \Omega_p \\ w &= u & \text{on } D_1 D \cup D_3 D \\ w &= 0 & \text{on } D_1 D_5 \cup D_3 D_4 \\ w &= N & \text{on } D_4 D_5 \\ w &= u & \text{in } B_1 \setminus \text{Sector } D_1 DD_3, \end{aligned}$$

where  $u$  is the one in Lemma 2.1.

We choose large enough  $N$  so that  $w_\gamma > u_\gamma$  on  $D_1 D \cup D_3 D$  and  $w_\gamma > 0$  on  $D_1 D_5 \cup D_3 D_4$ , where  $\gamma$  is the inward unit normal of  $\partial \Omega_p$  this time. (If one insists, we can smooth off  $\partial \Omega_p$ .)

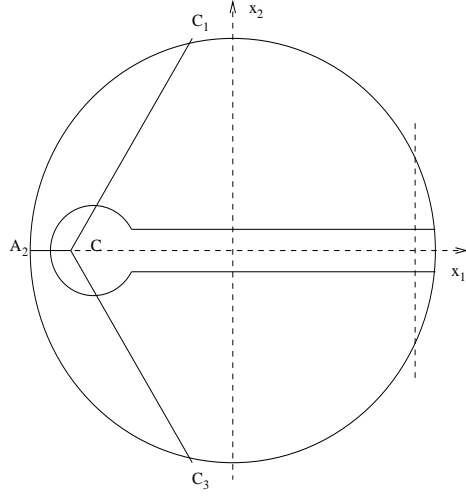


FIGURE 1. Minimal tree inside the half ball.

Next we mollify  $w$  by the usual (radially symmetric) mollifier  $\rho_\delta \in C_0^\infty(B_\delta)$  with  $0 < \delta < e^{-2K}$  to be determined later. We see that the smooth function  $w * \rho_\delta$  satisfies

$$\Delta w * \rho_\delta(x) \geq 0 \quad \text{for } x_1 \leq 19.9$$

$$w * \rho_\delta(x) = u \quad \text{for } x \text{ inside } \Omega_i = B_1 \setminus \text{Sector} D_1 D D_3 \text{ and } \delta \text{ away from } \partial\Omega_i$$

$$w * \rho_\delta(x) = 0 \quad \text{for } x \text{ outside } \Omega_o = (B_1 \setminus \text{Sector} D_1 D D_3) \cup \Omega_p \text{ and } \delta \text{ away from } \partial\Omega_o.$$

Finally, set  $C_0 = (-0.8, 0)$  and

$$v(x) = w * \rho_\delta(10(x - C_0)).$$

By making  $\delta$  even smaller yet positive if necessary so that  $\int_T v ds < 0$ , we obtain the desired function  $v$  in the above lemma.  $\square$

**Corollary 2.1.** *Let  $v$  be the function in Lemma 2.2. There exists a family of smooth metrics in  $\mathbb{R}^2$*

$$g_\delta = e^{2\delta v} dx^2 \quad \text{for } 0 < \delta < \delta_0$$

such that

$$g_\delta = dx^2 \quad \text{in } \{(x_1, x_2) \mid x_1 < 0.9\} \setminus B_1$$

$$K_{g_\delta} \leq 0 \quad \text{in } B_1$$

$$L(T, g_\delta) < L(T, dx^2),$$

where  $L(T, g)$  is the length of the minimal tree  $T$  from Lemma 2.2 in metric  $g$ .

*Proof.* We only prove the last two inequalities. One has

$$K_{g_\delta} = -e^{-2\delta v} \Delta(\delta v) \leq 0 \quad \text{in } B_1.$$

Also

$$L(T, g_\delta) = \int_T e^{\delta v} ds$$

$$\frac{dL}{d\delta} \Big|_{\delta=0} = \int_T v ds < 0.$$

Thus there exists  $\delta_0$  such that  $L(T, g_\delta) < L(T, dx^2)$  for  $0 < \delta < \delta_0$ .  $\square$

Let  $\psi \in C^1([-1, 1])$  satisfy  $0 \leq \psi \leq 1$  and  $\psi(\pm 1) = 0$ . Set  $\gamma = \{(x_1, x_2) \mid x_1 = \psi(x_2), |x_2| \leq 1\}$ ,  $Q = \{(x_1, x_2) \mid 0 < x_1 < \psi(x_2), |x_2| \leq 1\}$   $\Pi = [0, 2] \times [-2, 2] \subset \mathbb{R}^2$ ,  $F = \Pi \setminus Q$ .

**Lemma 2.3.** *Let  $f \in C^3(F)$ . Assume the graph  $\Sigma$  of  $f$  is flat or  $\det D^2 f = 0$  and  $D^2 f \neq 0$  in  $F$ . Also assume a unit  $C^1$  continuous eigenvector  $V_0$  for the zero eigenvalue of  $D^2 f$  is transversal to  $\gamma$ . For any  $0 < \tau < 1$ , there exists  $\varepsilon > 0$  so that if  $\left\| D^2 f - \begin{bmatrix} 0 & 0 \\ 0 & -\tau \end{bmatrix} \right\| \leq \varepsilon \tau$ , one can extend  $f$  to  $\Pi$  with the graph of the extension being flat and concave.*

*Proof.* We take the  $C^2$  Legendre coordinate system on  $F \subset \Pi$  (cf. [HW1]).

$$\begin{cases} t = x_1 \\ s = f_2(x_1, x_2). \end{cases}$$

Notice that the graph of  $f$ ,  $\Sigma$  is flat, or  $\det D^2 f = 0$ , it follows that  $\{(x_1, x_2) \mid f_2(x_1, x_2) = s = \text{const}\}$  is a straight segment in  $\mathbb{R}^2$  and  $x_t(t, s)$  ( $\|V_0\|$ ) is independent of  $t$ . Also  $\frac{\partial f}{\partial t}(x(t, s))$  is independent of  $t$ . Hence we can represent a portion  $\Sigma^p$  of the graph  $\Sigma$  in the ruling form

$$(x_1, x_2, x_3)(t, s) = h(t, s) = c(s) + t\delta(s) = (t, x_2(t, s), f(t, x_2(t, s))),$$

where  $c(s), \delta(s) \in C^2$  and  $s \in S = [f_2(2, 2), f_2(2, -2)]$ ,  $t \leq 2$ .

We may assume  $\nabla f(2, 0) = 0$ . If  $\varepsilon$  is chosen small enough, then  $\delta(s)$  ( $\|V_0\|$ ) is close to  $(1, 0, 0)$  in  $C^1$  norm. Take  $\varepsilon$  small, then

$$\{(x_1, x_2, f(x_1, x_2)) \mid ((x_1, x_2) \in \gamma)\} \subset \partial\Sigma^p.$$

Set  $U = \{(t, s) \mid -1 \leq t \leq 2, s \in S\}$ . Take  $\varepsilon$  small so that  $\|\delta(s) - (1, 0, 0)\|_{C^1}$  small, then  $(t, s) \in U$  is a  $C^2$  coordinate system for  $\Pi$ .

Now  $\Sigma^e = h(U)$  is a  $C^2$ , flat, concave graph over a domain  $\Omega$  in  $\mathbb{R}^2$  with  $\Pi \subset \Omega$ . Indeed, the normal of  $\Sigma^e$  is

$$N = \frac{h_t \times h_s}{\|h_t \times h_s\|}.$$

We know

$$h_t = \left( 1, \frac{-f_{21}}{f_{22}}, f_1 + f_2 \frac{-f_{21}}{f_{22}} \right) \xrightarrow{\varepsilon \rightarrow 0} (1, 0, 0)$$

$$h_s = \left( 0, \frac{1}{f_{22}}, \frac{f_2}{f_{22}} \right) \xrightarrow{\varepsilon \rightarrow 0} \left( 0, \frac{-1}{\tau}, \frac{-s}{\tau} \right),$$

then  $h_t \times h_s \xrightarrow{\varepsilon \rightarrow 0} (0, \frac{s}{\tau}, \frac{-1}{\tau})$ . So  $\Sigma^\varepsilon$  is a  $C^2$  graph if we choose  $\varepsilon$  small enough.

Next, the second fundamental form of  $\Sigma^\varepsilon$  is

$$\begin{aligned} II &= \begin{bmatrix} \langle h_{tt}, N \rangle & \langle h_{ts}, N \rangle \\ \langle h_{st}, N \rangle & \langle h_{ss}, N \rangle \end{bmatrix} \\ &= \frac{1}{\|h_t \times h_s\|} \begin{bmatrix} 0 & 0 \\ 0 & \langle c'' + t\delta'', \delta \times (c' + t\delta') \rangle \end{bmatrix} \end{aligned}$$

and the Gaussian curvature

$$K_g = 0.$$

Finally, the nonzero principle curvature of  $\Sigma^\varepsilon$

$$\kappa = \left[ \frac{\tau^3}{(1+s^2)^{3/2}} + o(\varepsilon) \right] \langle c'' + t\delta'', \delta \times (c' + t\delta') \rangle.$$

On the other hand, from the graph representation of  $\Sigma^p$ ,  $\kappa \xrightarrow{\varepsilon \rightarrow 0} -\tau / (1+s^2)^{3/2}$ . So for  $t$  in a certain range close to 2, say  $t \in [1, 2]$ , the quadratic function in terms of  $t$ ,

$$\langle c'' + t\delta'', \delta \times (c' + t\delta') \rangle = a_0 + a_1t + a_2t^2$$

is close to  $-1/\tau^2$  as  $\varepsilon \rightarrow 0$ . It follows that  $a_0 + a_1t + a_2t^2$  is still close to  $-1/\tau^2$  for  $t \in [-1, 2]$ , if we choose  $\varepsilon$  small enough. So  $\Sigma^\varepsilon$  is concave.  $\square$

**Lemma 2.4.** *Let  $f$  be the extended function in Lemma 2.3, let  $w \in C^2(\Pi)$  satisfy  $w = f$  on  $F$ ,  $\det D^2w \leq 0$  in  $\Pi$ , and  $\left\| D^2w - \begin{bmatrix} 0 & 0 \\ 0 & -\tau \end{bmatrix} \right\|_{C^1} \leq \varepsilon\tau$ . Then*

$$f \leq w \text{ in } \Pi.$$

*Proof.* Suppose there is a point  $x' = (x'_1, x'_2) \in M$  such that  $w(x') < f(x')$ . We know  $x'_2 \in (-1, 1)$ . For simplicity, we may assume

$$f(x') - w(x') = \sup_{x_2 \in [-1, 1]} [f(x'_1, x_2) - w(x'_1, x_2)].$$

Then  $f_2(x') = w_2(x')$ . It follows that the two tangent lines  $l_f, l_w$  to  $f$  and  $w$  at  $x'$  in the plane  $\{(x_1, x_2, x_3) | x_1 = x'_1\}$  are parallel. Since  $w(x'_1, \cdot)$  is concave,  $l_w$  is above  $w$ .

Let  $T \subset \mathbb{R}^3$  be the tangent plane to the graph  $\Sigma_f$  of  $f$  at  $(x', f(x'))$ . Let  $R = T \cap \Sigma_f$ . Then  $R$  is a segment (ruling) transversal to  $l_f$ . Let  $(x^0, z^0) \in R$  with  $x^0 \in F$ , then  $z^0 = f(x^0) = w(x^0)$ . Let  $l_0 \subset T$  through  $(x^0, z^0)$  with  $l_0 \parallel l_w$ . By the concavity of  $f = w$  in  $F$ ,  $l_0$  is above the graph  $\Sigma_w$  of  $w$ .

Let  $m(x)$  be the linear function with graph as the plane  $E$  through  $l_w$  and  $l_0$ . Let  $V = \{(x_1, x_2) | x'_1 < x_1 < 2, |x_2| < 2\}$ . Because  $\Sigma_w$  is a ruling surface on  $F$ , then

$$w(x) \leq m(x) \quad \text{on} \quad \partial V.$$

Note that  $\det D^2w \leq 0$ , by the maximum principle,

$$w(x) \leq m(x) \quad \text{in } V.$$

On the other hand, there is  $(x^*, w(x^*)) \in R$  with  $x^* \in V$  such that

$$w(x^*) > m(x^*).$$

This contradiction completes the proof of the above lemma.  $\square$

Let  $r$  be a rotation in  $\mathbb{R}^2$  through an angle  $1^\circ$ . Let  $v$  be the function in Lemma 2.2, set

$$w(x) = \sum_{i=1}^{360} v(r^i(1000x) - (360, 0)).$$

Pick two sequences  $z_n \in \mathbb{R}^2$  and  $\rho_n > 0$  such that

$$z_n \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

$$B_{\rho_n}(z_n) \cap B_{\rho_k}(z_k) = \emptyset \quad \text{for } n \neq k.$$

Take another sequence  $\delta_n > 0$  going to 0 fast enough so that the smooth metric  $g_{II}$  in  $\mathbb{R}^2$  satisfying

$$\begin{aligned} g_{II} &= e^{2\delta_n w(z_n + x/\rho_n)} dx^2 \quad \text{in } B_{\rho_n}(z_n) \\ g_{II} &= dx^2 \quad \text{otherwise.} \end{aligned}$$

**Remark.** Certainly our  $v$  is only smooth in  $B_{1.1}(0)$ , that leaves the function  $w$  nonsmooth, even undefined near the corresponding tails. At this stage, we do not need any information on the metric  $g_{II}$  near those tails (Figure 1 and 3). We can make a smooth extension of  $v$  to  $\mathbb{R}^2$  with  $v \in C_0^\infty(B_2)$  if one insists. Then the Gaussian curvature of  $g$  would be positive near the transition region. In the proof of Theorem 1.1, we will extend the tails to the boundary, make  $v$  a smooth subharmonic function inside the unit ball. Then the Gaussian curvature would be nonpositive in the unit ball.

**Proposition 2.1.** *Let  $i$  be a  $C^3$  isometric embedding*

$$i : (B_r(0), g_{II}) \rightarrow \mathbb{R}^3$$

*for some  $r > 0$ . Then the second fundamental form of  $i(B_r(0))$  vanishes at  $i(0)$ , or  $II(i(0)) = 0$ .*

*Proof.* We may assume  $i(B_r)$  is the graph  $\Sigma_w$  of a function  $x_3 = w(x_1, x_2)$  and  $w(0) = 0$ ,  $\nabla w(0) = 0$ . Then  $II(i(0)) = D^2w(0)$  and  $\det D^2w(0) = 0$ . Suppose

$$D^2w(0) \neq 0.$$

Let  $P_3$  be the projection from  $\mathbb{R}^3$  to  $x_1$ - $x_2$  plane. Set  $J(x) = P_3(i(x))$ . We may assume  $DJ$  is the identity map on the tangent space  $\mathbb{R}^2$  at 0, and

$$D^2w(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & -\tau \end{bmatrix}.$$



For a sufficiently large  $n$ ,  $B_{\rho_n}(z_n) \subset B_r$  and

$$g_{II} = e^{2\delta_n v(r^{180}(1000(z_n + x/\rho_n)) - (360, 0))} dx^2$$

in the  $179^\circ$  to  $181^\circ$  section of the ball  $B_{\rho_n}(z_n)$ .

In order to simplify the presentation, we work with the metric  $g_{\delta_n} = e^{2\delta_n v(x)} dx^2$  as in the Corollary 2.1. Let  $\Sigma^e$  be the flat, concave extension of  $i(B_2^- \setminus B_1^-)$  by Lemma 2.3, where  $B_\rho^- = \{(x_1, x_2) | x_1 < 0\} \cap B_\rho$ . Note that we may consider the graph  $x_3 = w_\varepsilon(x) = w(\varepsilon x)$  for small  $\varepsilon$ , then

$$\left\| D^2 w_\varepsilon - \begin{bmatrix} 0 & 0 \\ 0 & -\varepsilon^2 \tau \end{bmatrix} \right\|_{C^1} \leq \varepsilon^3,$$

make the extension, then scale back.

Since  $i(B_1^-)$  is negatively curved, or  $\det D^2 w \leq 0$  and concave, we apply Lemma 2.4 to conclude that  $i(B_1^-)$  is above  $\Sigma^e$ .

Let  $P$  be the normal projection of points  $p$  above  $\Sigma^e$  down to  $\Sigma^e$ , that is  $[p - P(p)] \perp \Sigma^e$ . By concavity of  $\Sigma^e$ , we have

$$\text{Length}(T, g_{\delta_n}) = \text{Length}(i(T), g_{\Sigma_w}) \geq \text{Length}(P(i(T)), g_{\Sigma^e}),$$

Where  $g_{\Sigma_w}$  and  $g_{\Sigma^e}$  is the induced metrics on  $\Sigma_w$  and  $\Sigma^e$ .

Note that  $P(i(C_1)) = i(C_1)$ ,  $P(i(C_3)) = i(C_3)$ ,  $P(i(A_2)) = i(A_2)$ , there is an isometry  $i_0 : \Sigma^e \rightarrow (\mathbb{R}^2, dx^2)$  such that  $i_0 \circ P \circ i(C_1) = C_1$ ,  $i_0 \circ P \circ i(C_3) = C_3$ ,  $i_0 \circ P \circ i(A_2) = A_2$ . Apply Corollary 2.1, we have

$$\text{Length}(P(i(T)), g_{\Sigma^e}) = \text{Length}(i_0 \circ P \circ i(T), dx^2) > \text{Length}(T, g_{\delta_n}).$$

Thus we arrive at

$$\text{Length}(T, g_{\delta_n}) > \text{Length}(T, g_{\delta_n}).$$

This contradiction finishes the proof of the above proposition.  $\square$

Now we give the constructive proof of Theorem 1.1.

*Proof.* Step1. Let  $\tilde{k}$  be a smooth function in  $\mathbb{R}^2$  satisfying

$$\begin{aligned} \tilde{k} &< 0 \quad \text{in } B^n = B_{2^{-2n}}(2^{-n}, 0), \quad n = 1, 2, 3, \dots \\ \tilde{k} &= 0 \quad \text{otherwise.} \end{aligned}$$

Let  $u_1$  be a smooth solution of

$$\Delta u_1 = -\tilde{k}.$$

Then the Gaussian curvature of the metric  $g_1 = e^{2u_1} dx^2$  satisfies

$$\begin{aligned} K_{g_1} &= -e^{-2u_1} \Delta u_1 < 0 \quad \text{in } B^n \\ K_{g_1} &= 0 \quad \text{otherwise.} \end{aligned}$$

Step2. Choose a sequence  $z_{n,k}$  outside each  $B^n$  and  $\{(x_1, x_2) | x_2 = 0\}$  such that

$$\lim_{k \rightarrow \infty} z_{n,k} \in \partial B^n$$

$$\partial B^n \subset \overline{\{z_{n,k}\}_{k=1}^{\infty}}.$$

For each  $z_{n,k}$ , choose a simply connected thin tail  $T_{n,k}$  with  $T_{n,k}$  connecting  $z_{n,k}$  and the boundary  $\partial B_1$  such that

$$z_{n,k} \in T_{n,k}$$

$$\partial T_{n,k} \cap \partial B_1 = \text{a piece of arc with positive length}$$

$$T_{n,k} \subset \mathbb{R}_+^2 = \{(x_1, x_2) | x_2 > 0\} \quad \text{for } x_2(z_{n,k}) > 0$$

$$T_{n,k} \subset \mathbb{R}_-^2 = \{(x_1, x_2) | x_2 < 0\} \quad \text{for } x_2(z_{n,k}) < 0$$

$$T_{n,k} \cap T_{m,j} = \emptyset \quad \text{for } (n, k) \neq (m, j).$$

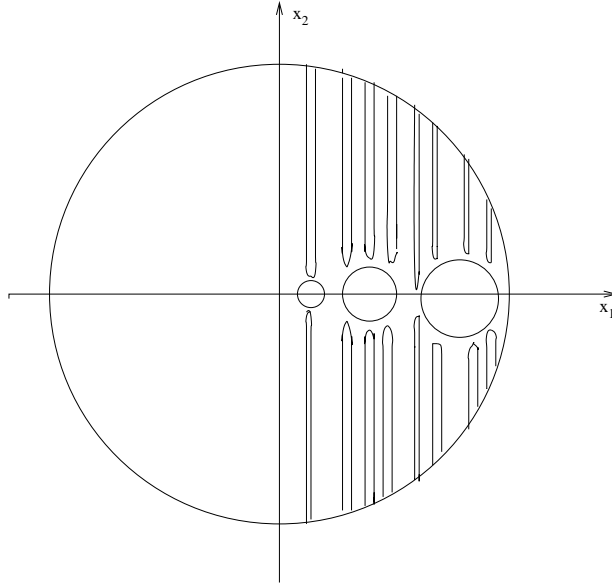


FIGURE 2. Tails extending to the boundary.

We modify the metric  $g_1 = e^{2u_1} dx^2$  over each tail  $T_{n,k}$ . But we proceed with the tails in the upper and lower half planes separately.

Since  $K_{g_1} \equiv 0$  in the simply connected domain  $\mathbb{R}_+^2 \setminus \cup_{n=1}^{\infty} B^n$ . We represent  $g_1 = dy_+^2$  in  $\mathbb{R}_+^2 \setminus \cup_{n=1}^{\infty} B^n$  by a different coordinate system  $y_+$ . Over each  $T_{n,k} \subset \mathbb{R}_+^2$ , we plant a metric

$$g_2 = e^{2V_{n,k}} dy_+^2 \quad \text{in } x^{-1}(T_{n,k}),$$

where  $V_{n,k}$  is similar to the one in the construction before Proposition 2.1, but the 360 disjoint sub-tails extend to the boundary  $x^{-1}(\partial B_1)$  within

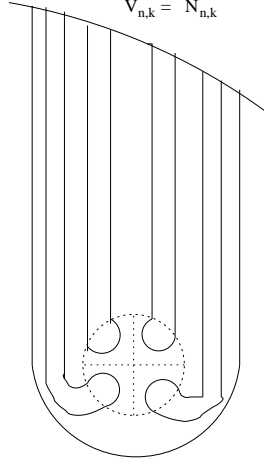


FIGURE 3. “Details” of one tail.

$x^{-1}(T_{n,k})$ . We know  $V_{n,k} = 0$  in  $x^{-1}(B_1 \setminus T_{n,k})$ . With  $V_{n,k} = N_{n,k}$  chosen large enough on  $x^{-1}(\partial B_1)$  intersection with the  $x$  pre-image of the 360 sub-tails, we make

$$\Delta V_{n,k} \geq 0 \quad \text{in} \quad x^{-1}(B_1).$$

We modify the metric  $g_1 = e^{2u_1} dx^2$  over the tails in the lower half plane  $\mathbb{R}_-^2$  with different coordinate system in the same way.

So far, we obtain a new metric  $g_2 = e^{2u_2} dx^2$  in  $B_1$  (which may not be smooth). We modify  $g_2$  over the tails one last time.

Let

$$\begin{aligned} g_3 &= e^{2\epsilon_{n,k} V_{n,k}} dy_+^2 & \text{in} \quad x^{-1}(T_{n,k}) & \text{for} \quad T_{n,k} \subset \mathbb{R}_+^2 \\ g_3 &= e^{2\epsilon_{n,k} V_{n,k}} dy_-^2 & \text{in} \quad x^{-1}(T_{n,k}) & \text{for} \quad T_{n,k} \subset \mathbb{R}_-^2. \end{aligned}$$

By choosing  $\epsilon_{n,k} > 0$ ,  $\epsilon_{n,k} \rightarrow 0$  sufficiently fast for  $k \rightarrow \infty$ , we can assure  $g_3 = e^{2u_3} dx^2$  is a smooth metric with  $K_{g_3} \leq 0$  in  $B_1$ .

Step 3. Suppose there is an isometric embedding

$$i : (B_r, g) \rightarrow \mathbb{R}^3$$

for some  $r > 0$ . Then there is  $n_*$  such that

$$B^{n_*} \subset B_r.$$

Applying Proposition 2.1, we have

$$II \circ i = 0 \quad \text{on} \quad \partial B^{n_*}.$$

We may assume  $i(B_r)$  is represented as a graph  $x_3 = f(x_1, x_2)$  with  $\nabla f(0, 0) = 0$ . Also we may assume the projection of  $i(B^{n_*})$  down to  $x_1$ - $x_2$  plane is a

domain  $\Omega$ . Then

$$\begin{aligned} \det D^2 f &= K_g \left(1 + |\nabla f|^2\right)^2 < 0 \quad \text{in } \Omega \\ D^2 f &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

From  $D^2 f = 0$  on  $\partial\Omega$ , it follows that  $\nabla f = \text{const.}$  on  $\partial\Omega$  and  $f$  coincides with a linear function on  $\partial\Omega$ . After subtracting the linear function from  $f$ , we may further assume  $f = 0$  on  $\partial\Omega$ . We still have  $\det D^2 f < 0$  in  $\Omega$ . From the maximum principle, we see that  $f \equiv 0$  in  $\Omega$ . This contradiction finishes the proof of Theorem 1.1.  $\square$

### 3. METRIC WITH NEGATIVE CURVATURE EXCEPT FOR ONE POINT

Relying on the metric constructed in Section 2, we construct a smooth metric  $g$  in  $B_1$  with negative Gaussian curvature except for one point, namely,  $K_g(x) < 0$  for  $x \neq 0$ , such that the surface  $(B_1, g)$  is not  $C^{3,\alpha}$  isometrically embeddable in  $\mathbb{R}^3$  even locally near 0.

For any surface  $(\Omega, g)$ , we define the  $C^{3,\alpha}$  isometric embedding norm by  $\|(\Omega, g)\|_E = \inf \{ \|II(i(\Omega))\|_{C^{1,\alpha}} \mid C^{3,\alpha} \text{ isometric embedding } i : (\Omega, g) \rightarrow \mathbb{R}^3 \}$ .

Now we give a constructive proof of Theorem 1.2.

*Proof.* Let the annulus  $A^n = B_{1/n} \setminus B_{1/(n+1)} \subset \mathbb{R}^2$ . We construct a metric  $g = e^{2u_0} dx^2$  on  $B_1$  such that a non-isometrically embeddable metric  $g$  as in Theorem 1.1 is planted (not just cut and pasted) over each annulus  $A^n$ .

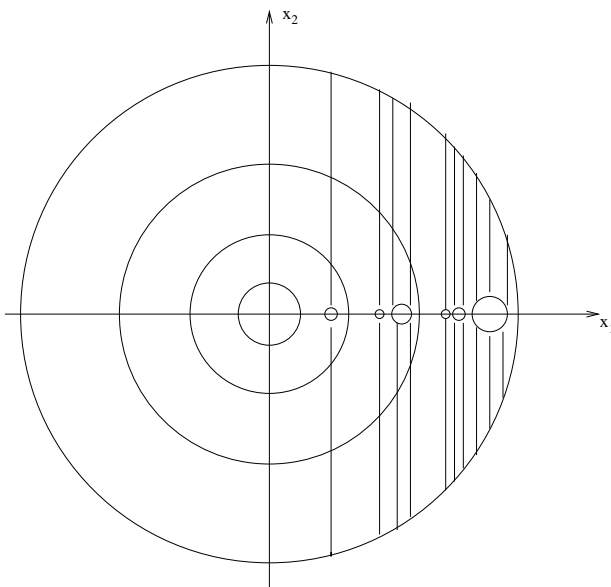


FIGURE 4. Non-embeddable metric in each annulus.

Set

$$\widetilde{\varphi}_n(r) = \begin{cases} e^{-\frac{1}{r-1/n}} & r = |x| > \frac{1}{n} \\ 0 & 0 \leq r \leq \frac{1}{n} \end{cases}$$

We choose  $\mu_1 > 0, \mu_2 > 0, \dots, \mu_n > 0, \dots$  such that  $\varphi_n = \mu_n \widetilde{\varphi}_n$  satisfies that  $\sum_{n=1}^{\infty} \varphi_n$  is smooth and even  $\sum_{n=1}^{\infty} \epsilon_n \varphi_n$  is smooth for  $(\epsilon_1, \epsilon_2, \dots) \in l^\infty$ .

For  $\epsilon = (\epsilon_1, \epsilon_2, \dots) \in l_+^\infty$ , that is  $\epsilon_1 > 0, \epsilon_2 > 0, \dots$  and  $\|\epsilon\|_\infty = \max \epsilon_m < +\infty$ , set

$$\Phi_\epsilon = \sum_{m=1}^{\infty} \epsilon_m \varphi_m$$

$$g_\nu = e^{2(u_0+\nu)} dx^2.$$

By the construction,  $(\mathbb{R}^n, e^{2u_0} dx^2)$  is not  $C^3$  isometrically embeddable in  $\mathbb{R}^3$  for any  $n$ , then we have the following.

There exists  $0 < \eta_1$  such that  $\|(A^1, g_{\Phi_\epsilon})\|_E \geq 1$  for  $\epsilon \in l_+^\infty$  with  $\|\epsilon\|_\infty \leq \eta_1$ .

Next there exists  $0 < \eta_2 < \eta_1$  such that  $\|(A^m, g_{\Phi_\epsilon})\|_E \geq m$  for  $m = 1, 2$  and  $\epsilon = (\eta_1, \epsilon_2, \epsilon_3, \dots) \in l_+^\infty$  with  $\|(0, \epsilon_2, \epsilon_3, \dots)\|_\infty \leq \eta_2$ .

Inductively there exists  $0 < \eta_k < \eta_{k-1}$  such that  $\|(A^m, g_{\Phi_\epsilon})\|_E \geq m$  for  $m = 1, 2, \dots, k$  and with  $\epsilon = (\eta_1, \eta_2, \dots, \eta_k, \epsilon_{k+1}, \epsilon_{k+2}, \dots) \in l_+^\infty$  with  $\|(0, \dots, 0, \epsilon_{k+1}, \epsilon_{k+2}, \dots)\|_\infty \leq \eta_k$ .

...

Finally let  $\Psi = \sum_{m=1}^{\infty} \eta_m \varphi_m$ ,  $g = g_\Psi$ . We see that

$$\|(A^m, g)\|_E \geq m \quad \text{for } m = 1, 2, 3, \dots$$

$$K_g(x) < 0 \quad \text{for } x \neq 0 \quad \text{and}$$

$$K_g(0) = 0.$$

It follows that there is no  $C^{3,\alpha}$  isometric embedding of  $(B_r(0), g)$  in  $\mathbb{R}^3$  for any  $r > 0, \alpha > 0$ .  $\square$

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