Chapter 4

Starting where you are

My initial objective is to consider a map, an actual geographical paper map or more properly a series of maps, charting locations in relation to Skiles classroom building (686 Cherry Street, Atlanta, GA 30332) starting with relatively nearby locations, say within twenty miles, and then working my way out. There is apparently readily available information on what are now called GPS coordinates[1] which can be thought of as a variant of spherical coordinates given in degrees as well as distances between various points on the earth. For example, I compiled the information in Table 4.1 in just a few minutes: Though it is not quite true, let us assume the first GPS coordinate is giving an angle of lattitude measured from the equator with 0° corresponding to the equator and positive angles measured between 0° and 90° in the northern hemisphere and angles measured between -90° and 0° in the southern hemisphere. The second GPS coordinate is giving an angle of rotation relative to the so-called prime meridian running through Greenwich, England with locations in North and South America having meridians predominantly corresponding to angles between -180° and 0°, most of Africa, Europe, Asia, and Australia have meridians with angles between 0° and 180°, and with Taveuni Island on the -180° meridian which is the same as the 180°

[1]Also known as **geodetic coordinates** these are technically somewhat complicated and dependent on a particular ellipsoidal approximation for the shape of the earth. I will briefly suggest some aspects of a possible ellipsoidal correction below. For our purposes, I hope we can assume the surface of the earth is spherical with a given constant radius, and make use of (only) spherical coordinates and an assumed relation between geodetic coordinates and sphereical coordinates. Technically, an approximation is swept under the rug in this relation.

Location	GPS azimuth	GPS meridian	distance
Skiles	33.7736	-84.396	0
big chicken	33.9515	-84.5208	14.25 mi.
Stone Mountain	33.811	-84.144	14.7 mi.
Wild Heaven Brewery	33.7306	-84.4194	3.26 mi.
Sweetwater Creek State Park	33.753	-84.63	13.45 mi.
Svalbard Museum†	78.22286	15.65251	4078.5
IMPA, Rio de Janeiro	-22.959	-43.2375	4763.36 mi

Table 4.1: Some locations along with distances to Skiles.

† temporarily closed

meridian.

Exercise 4.0.1 Find a location in North America with positive meridian angle.

Exercise 4.0.2 Find GPS coordinates for four locations within twenty-five miles of Atlanta, and determine the distances of those locations from Skiles classroom building.

Exercise 4.0.3 Make a version of the chart with the four nearby locations appearing in Table 4.1 similar to that illustrated in Figure 4.1.

(a) Add to your chart the four locations you considered in Exercise 4.0.2.

(b) Label all (9) locations, and indicate the distances from Skiles.

(c) Explain clearly the method and calculations involved in the construction of your chart.

Exercise 4.0.4 Using your chart and/or the reference information used to create your chart (but not using for example Google Maps or any other reference) calculate the six distances between pairs of locations you considered in Exercise 4.0.2.

Exercise 4.0.5 Explain clearly the method and calculations involved in your solution of Exercise 4.0.4.

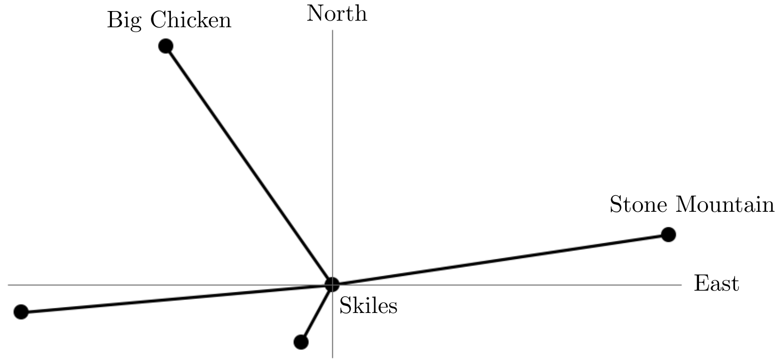

Figure 4.1: A map for navigation from Skiles classroom building

Exercise 4.0.6 Using a reference like Google Maps, check the distance values you obtained in Exercise 4.0.4.

Unfortunately, if one simply starts relatively nearby and attempts the mapping procedure as I had intended, then there seems to be some problem requiring some calibration as I will try to explain shortly. The issue of calibration accounts for the inclusion of the last two locations relatively far away (from Skiles). The radius of the earth is reported to have a minimum value $b = 3950$ miles (half the distance from the north pole to the south pole) and a maximum value $a = 3963$ miles at the equator. That is a difference of only 13 miles, and we may want to estimate how much that will effect our mapping distances. Alternatively, a better approximation depending on the azumuth ϕ of a particular location is given by

$$\sqrt{\frac{a^4 \cos^2 \phi + b^4 \sin^2 \phi}{a^2 \cos^2 \phi + b^2 \sin^2 \phi}}.$$

Denoting the radius of the earth by R (and assuming R is constant and the surface of the earth is spherical), the distance between two points with spherical coordinates (ϕ_1, θ_1) and (ϕ_2, θ_2) should be $R\theta$ where θ is the angle subtended at the center of the earth. The spatial locations of the two points with respect to the origin at the center of the earth are

$$\mathbf{p}_j = R(\sin \phi_j \cos \theta_j, \sin \phi_j \sin \theta_j, \cos \phi_j), \qquad j = 1, 2,$$

and the subtended angle θ should satisfy $\cos\theta = \langle \mathbf{p}_1, \mathbf{p}_2\rangle_{\mathbb{R}^3}/R^2$. If a subtended angle is obtained using the GPS coordinates, then notice this implies a radius $R = \text{distance}/\theta$ based on the distance given in Table 4.2. A calculation using mathematical software (Mathematica) gives the values in Table 4.2. Notice the variation in the implied radius in Table 4.2 among the

Location	$\cos\theta$	θ	implied R
big chicken	0.999994	0.00359333	3965.68 mi.
Stone Mountain	0.999993	0.00371302	3959.05 mi.
Wild Heaven Brewery	0.9999996607	0.000823739	3957.57 mi.
Sweetwater Creek State Park	0.999994	0.0033969	3959.47 mi.
Svalbard Museum[2]	0.514607	1.03251	3958.76
IMPA, Rio de Janeiro	0.359412	1.20316	3959.05 mi

Table 4.2: Angles subtended at the center of the earth relative to Skiles.

closer locations. All values fall within the advertised minimum $b = 3950$ mi. and maximum $a = 3963$ mi. except the one from the big chicken. Setting the big chicken aside, there is a variation of about 2 mi. among the values. Notice the variation arising from the last two far-flung locations is less than half a mile. The variation for the nearby locations is likely the result of the fact that the subtended angles corresponding to locations nearby Skiles are very small and the arccosine function is not differentiable at $\theta = 0$.

Exercise 4.0.7 Imagine there exist precise GPS coordinates (a_1, a_2) for some given point within the Skiles classroom building. Denote by (α_1, α_2) GPS coordinates of the Skiles classroom building that might appear in a table like Table 4.1 used to determine an implied radius of the earth as recorded in Table 4.2 in the discussion above. Similarly, denote by (b_1, b_2) a pair of precise GPS coordinates for one of the other locations in Table 4.1 and let (β_1, β_2) be the pair actually appearing in the table used to determine an implied radius of the earth. Finally, assume a precise distance c for the same location and a tabulated distance γ recorded for that location.

(a) Find reasonable but conservative[3] estimates for the errors in each of the tabulated values in Table 4.1:

[2] temporarily closed

[3] "Reasonable but conservative" means you are confident the error is no greater than this number, but you are also confident that this number is no greater than twice the error.

(i) Find $\epsilon_1 > 0$ and $\epsilon_2 > 0$ so that

$$|\alpha_1 - a_1| < \epsilon_1 \qquad \text{and} \qquad |\alpha_2 - a_2| < \epsilon_2.$$

(ii) Obtain reasonable but conservative azimuthal and meridian error estimates η_1 and η_2 respectivley for each of the alternative nearby locations in Table 4.1.

(iii) Obtain reasonable but conservative distance error estimates δ for each of the tabulated distances γ.

(b) Based on the actual values given in Table 4.1 and your reasonable but conservative error estimates, determine minimum and maximum possible values for the precise values:

(i) Given ϵ_1 and ϵ_2 find minimum and maximum possible values for a_j, $j = 1, 2$.

(ii) Given η_1 and η_2 for each location other than Skiles find minimum and maximum possible values for b_j, $j = 1, 2$ for each location.

(iii) Given δ for each location other than Skiles, find minimum and maximum values for c for each location other than Skiles.

(c) Based on the minimum and maximum values obtained in part **(b)** above, determine for each location ranges in which the tabulated values α_j, β_j, $j = 1, 2$ and γ must lie.

(d) Given GPS coordinate values (α_1, α_2) for Skiles and (β_1, β_2) for one of the locations other than Skiles along with a distance γ, and assuming all these values lie within the corresponding ranges determined in part **(c)**, quantify carefully the dependence of the **error** in the implied radius of the earth $R = R(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma)$ on the actual errors in each argument:

(i) If R_0 is the actual radius of the earth and R is the value determined by $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma)$ for a specific location, say Stone Mountain, find the maximum value of $|R - R_0|$ as a function of the increments $|\alpha_j - a_j| < \epsilon_j$, $|\beta_j - b_j| < \eta_j$ for $j = 1, 2$ and $|\gamma - c| < \delta$ over the ranges determined in part **(c)** for α, β, and γ.

(ii) Find the maximum error determined by all possible increments.

Based on the far-flung locations (IMPA and the Svalbard Museum which is temporarily closed) we take the radius of the earth to be $R = 3958.9$ miles. With this value I have created a small map as indicated in Figure 4.2.

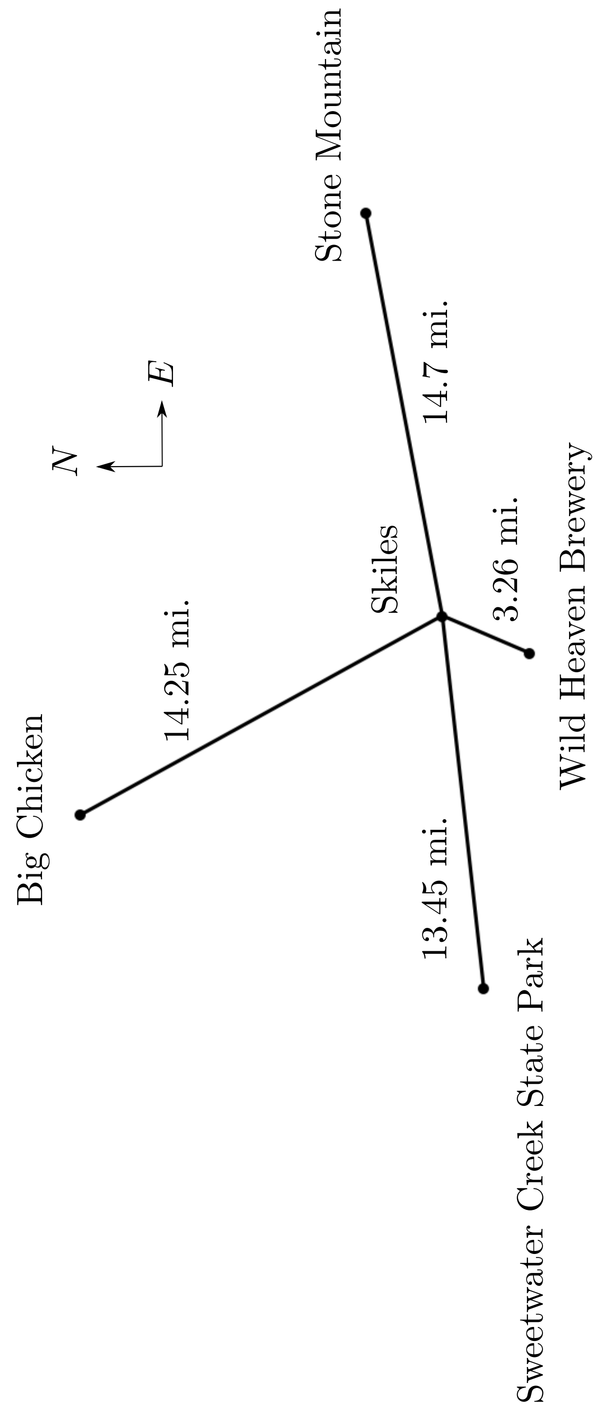

Figure 4.2: A map for navigation from Skiles classroom building