separation axioms and dimension of locally euclidean spaces

John Ioannis Stavroulakis

gt

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"The class of normal spaces is the weakest class of spaces where we define and study dimension functions. Although it is possible to define some kind of dimension for more general spaces, the result may be trivial and have little mathematical meaning."

K. Nagami
Dimension Theory, p. 44
Academic Press, 1970

regular

A topological space is *regular* if for every closed K every point $p \notin K$, there exist disjoint opens U, V such that $p \in U$ and $K \subset V$.

normal

A topological space is *normal* if for every pair of disjoint closed K, L, there exist disjoint opens U, V such that $K \subset U$ and $L \subset V$.

T_1

A topological space is T_1 if singletons are closed.

Hausdorff

A topological space is *Hausdorff* if for every pair of disjoint points p, q, there exist disjoint opens U, V such that $p \in U$ and $q \in V$.

ind

The (small) inductive (Brouwer-Menger-Urysohn) dimension of a topological space X, denoted by $\operatorname{ind} X \in \{-1, 0, 1, \ldots, +\infty\}$, is defined as satisfying

- $\operatorname{ind} X = -1$ iff $X = \emptyset$
- $\operatorname{ind} X \leq n$, where $n = 0, 1, \ldots$, if for every point $x \in X$ and every neighborhood $V \subset X$ of x there exists an open $U \subset V$ such that $x \in U$ and $\operatorname{ind} \partial U \leq n - 1$
- $\operatorname{ind} X = n$ if $\operatorname{ind} X \leq n$ and $\operatorname{ind} X > n - 1$
- $\operatorname{ind} X = \infty$ if $\operatorname{ind} X > n$ for $n = -1, 0, 1, 2, \ldots$

covering

The covering (also known as Čech-Lebesgue, or topological) dimension of a space X, denoted by $\dim X \in \{-1, 0, 1, 2, \ldots, \infty\}$, is defined by

- $\dim X \leq n \in \{-1, 0, 1, \ldots\}$ if every finite open cover of X has a finite open refinement of order n (the largest integer n such that the refinement contains $n+1$ sets with nonempty intersection)
- $\dim X = n$ if $\dim X \leq n$ and $\dim X > n-1$
- $\dim X = \infty$ if $\dim X > n$, for all $n = -1, 0, 1, \ldots$

Ind

The large inductive dimension of a topological space X, denoted by $\operatorname{Ind} X \in \{-1, 0, 1, ..., +\infty\}$, is defined as satisfying

- $\operatorname{Ind} X = -1$ iff $X = \emptyset$
- $\operatorname{Ind} X \leq n$, where $n = 0, 1, ...$, if for every closed $K \subset X$ and each open $V \subset X$ containing K, there exists an open $U \subset X$ such that $K \subset U \subset V$ and $\operatorname{Ind} \partial U \leq n - 1$
- $\operatorname{Ind} X = n$ if $\operatorname{Ind} X \leq n$ and $\operatorname{Ind} X > n - 1$
- $\operatorname{Ind} X = \infty$ if $\operatorname{Ind} X > n$ for $n = -1, 0, 1, 2, ...$

- R. Engelking
  Dimension Theory
  north-holland, 1978
- W. Hurewicz, H. Wallman
  Dimension Theory
  Princeton university press, 1941
- K. Nagami
  Dimension Theory
  Academic Press, 1970
- J. Nagata
  Modern Dimension Theory
  1965

Katětov-Morita

For any metrizable space X, we have $\operatorname{ind} X \leq \dim X = \operatorname{Ind} X$.

Theorem

For any separable metrizable space X, we have $\dim X = \operatorname{ind} X = \operatorname{Ind} X$.

Theorem

For any pair of compacta A, B, we have $\dim A \times B \leq \dim A + \dim B$.

Theorem

For any pair of subspaces of a metrizable space $A, B \subset C$, we have $\operatorname{Ind} A \cup B \leq \operatorname{Ind} A + \operatorname{Ind} B + 1$.

Fundamental theorem of dimension theory

$\dim \mathbb{R}^n = n$.

idea

Given a space locally homeomorphic to $\mathbb{R}^n$, show that the following implies the Hausdorff property: for any two points p, q, there exists an open set U containing both of them, with $\dim U = n$.

query

Can one characterize the Hausdorff property by dimension?

answer

Given that definitions of dimension functions typically require some separation properties in order to be interesting, sensical and nontrivial...probably not!

half-line with two origins

Consider A, B, C three copies of $(0, +\infty)$, along with p, q which intuitively correspond to the zero of A, B respectively. $A \cup p$ and $B \cup q$ are two copies of a half-line and C the rest of the real line (for both A and B). Define the opens to be the union of any sets in the original standard topology of A, B, C, along with all sets of the form $(0, \alpha) \cup p \cup (0, \gamma)$ (and corresponding sets with q).

Consider an open containing $\{p, q\}$, of the form

$$U := \{p, q\} \cup (0, \alpha) \cup (0, \beta) \cup (0, \gamma),$$

and a (finite) open cover Θ of U. As Θ is open, it contains a set of the form

$$V = \{p\} \cup (0, \alpha) \cup (0, \gamma),$$

and

$$W = \{q\} \cup (0, \beta) \cup (0, \gamma'),$$

where $\gamma' \leq \gamma$. We may replace W with

$$Y = \{q\} \cup (0, \beta) \cup (0, \frac{1}{2}\gamma')$$