Final Problems
Riemannian geometry

MATH 6455, spring semester 2024

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Problem 1 Let M denote the real line as a topological manifold. Consider the global charts

(i) $\mathbf{p} : \mathbb{R}^1 \to M$ by $\mathbf{p}(x) = x^3$,

(ii) $\mathbf{q} : \mathbb{R}^1 \to M$ by $\mathbf{q}(x) = x^{1/3}$, and

(iii) $\text{id} : \mathbb{R}^1 \to M$ by $\text{id}(x) = x$.

Let $\mathcal{A}_*$ denote the maximal C^∞ atlas for M containing $\text{id} : \mathbb{R}^1 \to M$.

(a) Is $\mathbf{p} \in \mathcal{A}_*$?

(b) Is $\mathbf{q} \in \mathcal{A}_*$?

Problem 2 Let M, $\mathbf{p}$, $\mathbf{q}$ and $\mathcal{A}_*$ be as in Problem 1.

(i) Let $\mathcal{A}_*^P$ denote the maximal C^∞ atlas for M containing $\mathbf{p} : \mathbb{R}^1 \to M$.

(ii) Let $\mathcal{A}_*^q$ denote the maximal C^∞ atlas for M containing $\mathbf{q} : \mathbb{R}^1 \to M$.

What can you say about the (relations among) the sets $\mathcal{A}_*$, $\mathcal{A}_*^p$ and $\mathcal{A}_*^q$?

Problem 3 Let M, $\mathbf{p}$, $\mathbf{q}$, $\mathcal{A}_*$, $\mathcal{A}_*^p$ and $\mathcal{A}_*^q$ be as in Problem 2.

(i) Let $\mathbb{R}$ denote the Riemannian manifold $(M, \mathcal{A}_*)$ with the standard inner product given by multiplication.

(ii) Let $M(p)$ denote any Riemannian manifold $(M, \mathcal{A}_*^p)$ determined by some metric tensor μ.

A function $\psi : M(p) \to \mathbb{R}$ for which $\psi \circ \mathbf{p} \in C^\infty(\mathbb{R}^1)$ is a diffeomorphism, i.e., $\psi^{-1} : \mathbb{R} \to M(p)$ exists and $(\psi \circ \mathbf{p})^{-1} \in C^\infty(\mathbb{R}^1)$, is an **isometry** if

$$(\psi \circ \alpha)'(t_1)(\psi \circ \beta)'(t_2) = \mu_x([\alpha], [\beta])$$

for all $\alpha, \beta \in c\mathfrak{I}^\infty(M(p))$ with $\alpha(t_1) = \beta(t_2) = x \in M(p)$. Recall that $c\mathfrak{I}^\infty(M(p))$ denotes the chart smooth embedded paths in $M(p)$ so that t_1 and t_2 are well-defined.

Characterize the metrics μ for which there exists an isometry $\psi : M(p) \to \mathbb{R}$.

Problem 4 Given Riemannian manifolds M and N with maximal C^∞ atlases $\mathcal{A}_*$ and $\mathcal{B}_*$ and Riemannian metric tensors μ and ν respectively, a function $\psi : M \to N$ is an **isometry** if

(i)
$$\eta \circ \psi \circ \mathbf{p}\big|_{\xi(W)} \in C^\infty(\xi(W))$$
whenever $\mathbf{p} : U \to M$ and $\mathbf{q} : V \to N$ are chart functions in $\mathcal{A}_*$ and $\mathcal{B}_*$ respectively with inverses $\xi : \mathbf{p}(U) \to U$ and $\eta : \mathbf{q}(V) \to V$ and for which $W = \mathbf{p}(U) \cap \mathbf{p}(V) \neq \phi$.

(ii) $\psi^{-1} : N \to M$ exists and
$$\xi \circ \psi^{-1} \circ \mathbf{q}\big|_{\eta(W)} \in C^\infty(\eta(W))$$
whenever $\mathbf{p} : U \to M$ and $\mathbf{q} : V \to N$ are chart functions in $\mathcal{A}_*$ and $\mathcal{B}_*$ respectively with inverses $\xi : \mathbf{p}(U) \to U$ and $\eta : \mathbf{q}(V) \to V$ and for which $W = \mathbf{p}(U) \cap \mathbf{p}(V) \neq \phi$.

(iii)
$$\mu_P([\alpha], [\beta]) = \nu_{\psi(P)}([\psi \circ \alpha], [\psi \circ \beta])$$
for all $\alpha, \beta \in c\mathfrak{I}^\infty(M)$ with $\alpha(t_1) = \beta(t_2) = P \in M$.

Let M be a connected one dimensional Riemannian manifold with Riemannian metric tensor μ. Characterize all the possible isometry classes for M, i.e., give a family $\mathcal{M}$ of "standard" one dimensional Riemannian manifolds like $\mathbb{R}$ and $\mathbb{S}^1$ such that not two are isometric and every M is isometric to one of the manifolds in $\mathcal{M}$.

Problem 5 Calculate the Riemannian curvature operator and the Ricci curvature operator on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ for $n \geq 4$. In this problem $\mathbb{S}^n$ is considered with the Riemannian metric tensor induced on $\mathbb{S}^n$ as a submanifoled of the ambient $\mathbb{R}^{n+1}$.

Problem 6 (Challenge) Consider the one parameter family of Riemannian metric tensors $\mu(t)$ on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ satisfying
$$\frac{d}{dt}\mu(t) = \mathrm{Ric}^{M(t)} \qquad \text{with} \qquad \mu(0) = \langle\ \cdot\ ,\ \cdot\ \rangle_{\mathbb{R}^{n+1}}. \tag{1}$$
Describe the resulting Riemannian manifolds $M(t)$. For this problem $M(0)$ is $\mathbb{S}^n$ considered with the Riemannian metric tensor induced on $\mathbb{S}^n$ as a submanifoled of the ambient $\mathbb{R}^{n+1}$, but $M(t)$ is $\mathbb{S}^n$ considered as a Riemannian manifold with the metric tensor $\mu(t)$ determined by the initial value problem (1). Hint: Write (1) down locally in terms of a chart $\mathbf{p} : U \to \mathbb{S}^n$ where U is a ball in $\mathbb{R}^n$ and the equation takes the form
$$\frac{d}{dt} g_{ij} = \mathrm{Ric}^{M(t)}_{ij}$$
where $g_{ij} = \mu(v_i, v_j)$ and $\mathrm{Ric}_{ij} = \mathrm{Ric}(v_i, v_j)$ are real valued functions on U. This gives Hamilton's Ricci flow on a sphere.