Chapter 13

Spaces and spaces of functions

When we speak of a "space," we usually mean a set with some kind of "structure" which can be assumed among and between the elements of the set. We have mentioned such structures above and we will mention more below. Some structures are fundamentally algebraic, like structures associated with addition and multiplication or scaling. These give rise to, for example, linear spaces.[1] Some structures are also algebraic and more elementary. In this category are algebraic groups which will be discussed below. Other structures are fundamentally "analytic" like those associated with a norm or an inner product, and possibly even a distance function giving rise to normed spaces, inner product spaces, and distance (a.k.a. metric) spaces, all of which will also be discussed below. Some structures do not fundamentally concern individual elements in the set but rather subsets and collections of subsets. Two examples of these kinds of "spaces" are measure spaces and topological spaces. We begin our discussion with a brief introduction to the latter. Many of the spaces we discuss are isolated according to properties relatively familiar from the consideration of $\mathbb{R}^n$ and the calculus outlined in the previous chapter.

13.1 Topological spaces

A **topological space** is a set X with a specified collection $\mathfrak{T}$ of subsets of X called the **topology** on X or the **collection of open sets**. A topology is a collection of open sets. More generally, the collection of all subsets of a

[1] We attempt to distinguish here between linear spaces and vector spaces.

given set X is called the **power set** of X and is denoted $\wp(X)$ or 2^X. Thus, a topology $\mathfrak{T}$ is a subset of $\wp(X)$. Specifically, a collection $\mathfrak{T}$ qualifies as a collection of open sets if the following hold:

(i) $\phi, X \in \mathfrak{T}$.

(ii) If $U_1, U_2, \ldots, U_k \in \mathfrak{T}$ for some $k \in \mathbb{N}$, then

$$\bigcap_{j=1}^{k} U_j \in \mathfrak{T}.$$

We say a topology $\mathfrak{T}$ is **closed under finite intersections**.

(iii) If $U_\alpha \in \mathfrak{T}$ for α in some indexing set Γ, then

$$\bigcup_{\alpha \in \Gamma} U_\alpha \in \mathfrak{T}.$$

We say a topolgy $\mathfrak{T}$ is **closed under arbitrary unions.**

It is nice to know about topological spaces in this generality, and it is nice to have a working knowledge of some characteristic properties of such spaces. We will discuss some of these topics presently. It is not always so nice to have to work with a specific topological space if it does not have some additional structure beyond the general definition. Very specifically, we will almost always assume any topological space under consideration satisfies the following condition concerning individual elements:

> A topological space X, with topology $\mathfrak{T}$, is said to be **Hausdorff** if given any $x, y \in X$ with $x \neq y$, there exist open sets $U_1, U_2 \in \mathfrak{T}$ for which
>
> $$x \in U_1, \qquad y \in U_2, \qquad \text{and} \qquad U_1 \cap U_2 = \phi.$$

The points x and y are said to be **separated** by the open sets U_1 and U_2.

As mentioned above, a topological structure can be isolated in $\mathbb{R}^n$. The collection of open sets $\mathfrak{T}_n$ in $\mathbb{R}^n$ is the collection of all unions of open balls:

$$\mathfrak{T}_n = \left\{ U = \bigcup_{\alpha \in \Gamma} B_{r_\alpha}(\mathbf{x}_\alpha) \in 2^X : r_\alpha > 0 \text{ and } \mathbf{x}_\alpha \in \mathbb{R}^n \text{ for } \alpha \in \Gamma \right\} \tag{13.1}$$

where Γ is some index(ing) set and

$$B_r(\mathbf{p}) = \left\{ \mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sqrt{\sum_{j=1}^{n} (x_j - p_j)^2} < r \right\} \tag{13.2}$$

for $r > 0$ and $\mathbf{p} = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n$. Note carefully, I do not mean to say Γ is a fixed indexing set determined outside the expression/definition (13.1) but rather the symbol Γ as it appears in (13.1) simply denotes some set used to index the centers $\mathbf{x}_\alpha$ and radii r_α used to construct a particular open set $U \in \mathfrak{T}_n$.

Exercise 13.1. Show $\mathfrak{T}_n$ is a topology on $\mathbb{R}^n$.

Exercise 13.2. Show $\mathfrak{T}_n$ is Hausdorff.

If you've never seen the definition of a topological space given above (and you are reading these notes carefully and critically) then you might be troubled that according to the definition, the "topology" is supposed to tell you which sets are **open**, but the topology for $\mathbb{R}^n$ is described in terms of "open" balls. The resolution is easy: The first use of "open" in regard to the "open" balls defined in (13.2) is just an informal use of the word and can be freely omitted. But then, according to the definition (and the assertion of Exercise 13.1) the sets defined in (13.2) are immediately seen to be open in the technical sense of the definition. In fact, each open ball is the union of the single open ball which is itself. Thus, every open ball is indeed an open ball. Sometimes the open sets in $\mathbb{R}^n$ are introduced in a slightly different way: A set $A \subset \mathbb{R}^n$ is **open** if for each $\mathbf{p} \in A$, there is some $r > 0$ such that $B_r(\mathbf{p}) \subset A$, where $B_r(\mathbf{p})$ is defined just as in (13.2). This leads to the following standard exercise which is a little more interesting and should be familiar to those who have seen topological spaces before.

Exercise 13.3. Show an open ball $B_r(\mathbf{p})$ in $\mathbb{R}^n$ is open.

For those to whom Exercise 13.3 is old news, I offer Exercise 13.7 below.

When all spaces under consideration share the same structure and the focus of the discussion is primarily related to that specific structure, one sometimes refers to the ensuing discussion as concerning the **category** of such spaces. For example, this section in my notes is about (or takes place in) the category[2] of topological spaces.

There is another homonymous use of words in the definition of topological space above. Being "closed" under unions or intersections is altogether entirely different from being topologically closed.

Definition 17. A subset A of a topological space X is said to be **closed** if the set $X \backslash A = \{x \in X : x \notin A\}$, i.e., the complement of A is open.

Again, if one has not encountered topological spaces of this sort (in this generality or specifically using the definition above) there are at least a few exercises one should do concerning the basics of open and closed sets. I will try to remember and include some of them before this section is done. Here is one:

Exercise 13.4. Let X be a topological space.

(a) Show an arbitrary intersection of closed sets is closed.

(b) Show a union of finitely many closed sets is closed.

(c) Show ϕ and X are closed.

(d) Give an example of a union of closed sets which is open and not closed.

(e) Try to find a proper subset of $\mathbb{R}$ (with the usual topology defined/described above) which is not ϕ and is not $\mathbb{R}$, but is both open and closed.

[2]This is an informal usage of the word category. There is a formal mathematical subject called category theory in which axioms defining exactly what is meant by a category are laid down, and one attempts to pursue a discussion in the category of categories. It may be presumed even from this phrase, that the subject is probbably of somewhat limited use. In any case, I have no use for category theory. The informal use of the term "category" however, and sometimes related terminology from category theory, can be suggestive and convenient.

Continuous functions

The main distinctive of topological spaces is that if you have two of them, say X and Y, then it makes sense to talk about continuous functions

$$f : X \to Y$$

and the "space" $C^0(X \to Y)$ of all such continuous functions.[3] Notice that $C^0(X \to Y)$ is not a vector space or anything like that in this generality, because there is no albebraic structure in sight here. What we do need, however, is a definition.

Definition 18. Given topological spaces X and Y and a function $f \in Y^X$, we say f is **continuous** and write $f \in C^0(X \to Y)$ if

$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$

is open in X whenever V is an open set in Y.

For functions $f : \mathbb{R} \to \mathbb{R}$ (or $f : \mathbb{R}^n \to \mathbb{R}^m$ or various related functions) one usually introduces **continuity at a point** first, and the definition is probably familiar: $f : \mathbb{R} \to \mathbb{R}$ is continuous at $p \in \mathbb{R}$ if for any $\epsilon > 0$, there is some $\delta > 0$ for which

$$f(B_\delta(p)) = \{f(x) : |x - p| < \delta\} \subset B_\epsilon(f(p)).$$

The way I have written this should be fairly suggestive of how the definition might look in more general situations. There are a bunch of at least nominally interesting theorems saying things like: If f and g are continuous at a point $p \in \mathbb{R}$, then $f + g$ is continuous at p. These kinds of theorems do not appear in the category of topological spaces because, as mentioned above, you need algebraic structure (at least on the codomain) to make sense of $f + g : X \to Y$, so it is simply that the function $f + g$ does not really appear.

You can make a definition of pointwise continuity for a function $f : X \to Y$ from one general topological space X to another Y, but then you do not have any particularly interesting theorems to go with that definition (as far as I know). If you know of such a theorem, let me know.

Returning to $f : \mathbb{R} \to \mathbb{R}$, one says f is continuous on $\mathbb{R}$ if f is continuous at each point $p \in \mathbb{R}$. Now we have two nominally different looking definitions of continuity.

[3] One might say the "morphisms" in the category of topological spaces are (the) continuous maps, were one inclined to say such things.

Exercise 13.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) Show that if for each $p \in \mathbb{R}$ and each $\epsilon > 0$, there is some $\delta > 0$ for which

$$f(B_\delta(p)) = \{f(x) : |x - p| < \delta\} \subset B_\epsilon(f(p)),$$

then the inverse image of every open set is open, that is

$$f^{-1}(V) = \{x \in \mathbb{R} : f(x) \in V\}$$

is open whenever V is an open subset of $\mathbb{R}$.

(b) Show conversely, that if the inverse image of every open set is open, then f is continuous at each point $p \in \mathbb{R}$.

Induced topologies

There are various general ways to create new topological spaces from topological spaces you have in hand. Two of the most important involve what are called the **subspace topology** and the **product topology**.

Exercise 13.6. Given any topological space X and any subset $A \subset X$, show the set

$$\{A \cap U : U \in \mathfrak{T}\}$$

where $\mathfrak{T}$ is the topology on X is a topology on A. This topology is called the **subspace topology** on A.

Exercise 13.7. Let A be a subset of $\mathbb{R}^n$ considered as a topological subspace of $\mathbb{R}^n$ with respect to the subspace topology. Show A is an open subset of $\mathbb{R}^n$ if and only if every open set in A can be written as a union of open balls $B_r(\mathbf{p}) \subset A$.

Perhaps the most important thing about the subspace topology is that the construction is valid for absolutely **any** subset. The subspace A is not required to be open, closed, or anything else. As a consequence, we can freely talk about continuous functions on any subset of $\mathbb{R}$ or $\mathbb{R}^n$. In particular, the real valued functions on a set $A \subset \mathbb{R}^n$ denoted by $C^0(A)$ are well-defined using the subspace topology on A. This kind of thing often doesn't work out so well if one is using some other structure beyond continuity to specify functions. For example, if one wants to talk about derivatives of a function

$f : A \to \mathbb{R}$ which typically involves the consideration of **increments** at some level, that is, expressions of the form

$$f(\mathbf{p} + \mathbf{v}) - f(\mathbf{p}),$$

then it makes things really convenient if A is open so that $\mathbf{p} + \mathbf{v} \in A$ when the increment $\mathbf{v} = (\mathbf{p} + \mathbf{v}) - \mathbf{p}$ is small.

The **product topology** gives a topology on a set

$$\prod_{j=1}^{n} X_j = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) : x_j \in X_j, \text{ for } j = 1, 2, \ldots, n\}$$

where X_j is a topological space with topology $\mathfrak{T}_j$ for $j = 1, 2, \ldots, n$. This should look familiar, and a conflict or inconsistency in notation should be noticed. I used the notation $\mathfrak{T}_n$ to denote the topology on $\mathbb{R}^n$. Here the same symbol is used in connection with arbitrary topological spaces X_j for $j = 1, 2, \ldots, n$.

Exercise 13.8. Given topological spaces X_j for $j = 1, 2, \ldots, n$ with topologies $\mathfrak{T}_j$ as described above, show the collection of unions of **open cubes**

$$\left\{ U = \bigcup_{\alpha \in \Gamma} C_\alpha : C_\alpha = \prod_{j=1}^{n} V_{\alpha,j}, \ V_{\alpha,j} \in \mathfrak{T}_j, \ \alpha \in \Gamma \right\}$$

is a topology where

$$\prod_{j=1}^{n} V_j = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) : x_j \in V_j \text{ for } j = 1, 2, \ldots, n\}.$$

The convention concerning the index set Γ is similar to the one used in regard to (13.1) above.

The topology defined in Exercise 13.8 is called the **product topology**. The definition we have given works fine for finitely many factors (i.e., factor spaces X_j). For now, we will attempt to avoid consideration of larger products

$$\prod_{\alpha \in \Gamma} X_\alpha$$

where Γ is an infinite set and there are infinitely many topological spaces X_α. For infinitely many factors the situation becomes more complicated.

Exercise 13.9. Show that if one takes $X_j = \mathbb{R}$ for $j = 1, 2, \ldots, n$ in the definition of the product topology in Exercise 13.8, then one gets the ball topology defined on $\mathbb{R}^n$ at the beginning of this section.

Homeomorphism

Given a bijection $f \in C^0(X \to Y)$ where X and Y are topological spaces, there is an inverse function $f^{-1} : Y \to X$, but it is not always true that this function is continuous.

Exercise 13.10. Consider $f : [0, 2\pi) \to \mathbb{S}^1$ by

$$f(x) = (\cos t, \sin t)$$

where $[0, 2\pi) \subset \mathbb{R}$ and

$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$$

are considered in the subspace topology induced by the topologies on $\mathbb{R}^1$ and $\mathbb{R}^2$ respectively.

(a) Show $f \in C^0([0, 2\pi) \to \mathbb{S}^1)$.

(b) Show f is a bijection.

(c) Show $f^{-1} \notin C^0(\mathbb{S}^1 \to [0, 2\pi))$.

Definition 19. If X and Y are topological spaces and $f \in C^0(X \to Y)$ is a bijection **and** $f^{-1} \in C^0(Y \to X)$, then we say f is a **homeomorphism**.

Exercise 13.11. If X and Y are topological spaces and $f \in C^0(X \to Y)$ is a bijection **and**

$$f(U) = \{f(x) : x \in U\} \qquad \text{is open in } Y \text{ for every } U \text{ open in } X,$$

then f is a homeomorphism.

A function $f \in C^0(X \to Y)$ for which

$$f(U) = \{f(x) : x \in U\} \qquad \text{is open in } Y \text{ for every } U \text{ open in } X$$

is called an **open mapping**.

Exercise 13.12. Find an example of an open mapping which is not a homeomorphism.

Topology induced by a bijection

Exercise 13.13. If X is a topological space and $f : X \to Y$ is a bijection, then show the following:

(a) $\mathfrak{T}_Y = \{f(U) : U \text{ is open in X}\}$ is a topology on Y.

(b) Using $\mathfrak{T}_Y$ to make Y a topological space, f is a homeomorphism.

(c) If X is a Hausdorff space, then the topology induced by f on Y is a Hausdorff topology.

Connectedness and Compactness

Two mathematical concepts which are important for Riemannian geometry and analysis in general are the concepts of **connectedness** and **compactness**. These are essentially topological concepts, so it makes sense to introduce them here.

Connectedness

The idea of a set being connected may be viewed as closely related to the Hausdorff condition. The Hausdorff condition requires that two points can be separated by open sets. More generally, two sets A and B with $A, B \neq \phi$ can be **separated** if there exist open sets U_1, U_2 for which

$$A \subset U_1, \qquad B \subset U_2, \qquad \text{and} \qquad U_1 \cap U_2 = \phi.$$

A set C in a topological space X is **connected** if C cannot be partitoned into nonempty sets that can be separated. That is, C is connected if whenever $C = A \cup B$ with $A, B \neq \phi$, then A and B cannot be separated.

Exercise 13.14. Let C be a subset of a topological space X considered with respect to the subspace topology on C. Show C is connected if and only if whenever $C = U_1 \cup U_2$ for U_1 and U_2 disjoint open sets in C, then $U_1 = \phi$ or $U_2 = \phi$.

Exercise 13.15. Let C be a connected topological space. Show that if $A \subset C$ is both open and closed, then $A = C$ or $A = \phi$.

Exercise 13.16. A **connecting path** in a topological space X is a continuous function $\alpha : I \to X$ where $I = [a, b] \subset \mathbb{R}$ for some $a, b \in \mathbb{R}$ wtih $a < b$. We say a connecting path connects the points $x = \alpha(a)$ and $y = \alpha(b)$, and a topological space X is **path connected** if for every pair of points $x, y \in X$ there exists a connecting path connecting x and y.

(a) Show a path connected space is connected.

(b) Given an example of a connected space which is not path connected.

(c) Show every connected open subset of $\mathbb{R}^n$ is path connected.

Exercise 13.17. The continuous image of a connected set is connected. That is, if X and Y are topological spaces, $f \in C^0(X \to Y)$, and $C \subset X$ is connected, then

$$f(C) = \{f(x) : x \in C\}$$

is connected.

Simply connected spaces

Let X be a path connected topological space, and consider two paths $\alpha : [a, b] \to X$ and $\beta : [a, b] \to X$ defined on the same interval $I = [a, b] \subset \mathbb{R}$ for some $a, b \in \mathbb{R}$ with $a < b$ and connecting the same points $x = \alpha(a) = \beta(a)$ and $y = \alpha(b) = \beta(b)$. A **homotopy** deforming α to β is a function $h \in C^0([a, b] \times [0, 1] \to X)$ satisfying the following:

(i) $h(s, 0) \equiv \alpha(t)$ for $s \in [a, b]$.

(ii) $h(s, 1) \equiv \beta(t)$ for $s \in [a, b]$.

(iii) $h(a, t) \equiv x$ for $t \in [0, 1]$.

(iv) $h(b, t) \equiv y$ for $t \in [0, 1]$.

Technically, this might be called a **fixed endpoint homotopy**, but I think this is mostly the only kind of homotopy we need to consider at the moment.

Exercise 13.18. Consider

$$X = B_5(\mathbf{0}) \backslash \overline{B_1(\mathbf{0})} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : 1 < x_1^2 + x_2^2 < 25\}$$

and the paths $\alpha : [0, 2\pi] \to X$ by $\alpha(s) = (4 \cos s, 4 \sin s)$ and $\beta : [0, 2\pi] \to X$ by

$$\beta(s) = (3 + \cos s, \sin s).$$

(a) Find a function $h \in C^0([0, 2\pi] \times [0, 1] \to X)$ satisfying

- **(i)** $h(s, 0) \equiv \alpha(t)$ for $s \in [a, b]$.
- **(ii)** $h(s, 1) \equiv \beta(t)$ for $s \in [a, b]$.

(a) Find a function $g \in C^0([0, 2\pi] \times [0, 1] \to X)$ satisfying

- **(i)** $g(s, 0) \equiv \alpha(t)$ for $s \in [a, b]$.
- **(ii)** $g(s, 1) \equiv (4, 0)$ for $s \in [a, b]$.

Note: The functions h and g in this exercise are **not** homotopies according to the definiton given above.

A path connected topological space X is said to be **simply connected** if given any path $\alpha \in C^0([a, b] \to X)$ where $[a, b]$ is an interval as in the definition of a homotopy above and $\alpha(a) = \alpha(b)$, there exists a homotopy $h \in C^0([a, b] \times [0, 1] \to X)$ for which $h(s, 0) \equiv \alpha(s)$ for $s \in [a, b]$ and $h(s, 1) \equiv \alpha(a)$ for $s \in [a, b]$.

Exercise 13.19. Show

$$\mathbb{S}^{n-1} = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\} \subset \mathbb{R}^n$$

is simply connected for $n \geq 3$.

The Poincaré conjecture discussed in Chapter 7 is a conjecture about certain simply connected topological spaces.

Compactness

A collection of sets $\{A_\alpha\}_{\alpha \in \Gamma}$ where Γ is some indexing set is said to **cover** a set A if

$$\bigcup_{\alpha \in \Gamma} A_\alpha \supset A.$$

In the case where X is a topological space with topology $\mathfrak{T}$, we have a set $A \subset X$, and $\{U_\alpha\}_{\alpha \in \Gamma} \subset \mathfrak{T}$ satisfies

$$\bigcup_{\alpha \in \Gamma} U_\alpha \supset A$$

we say $\{U_\alpha\}_{\alpha \in \Gamma}$ is an **open cover** of A.

A set K in a topological space X is said to be **compact** if given any open cover $\{U_\alpha\}_{\alpha\in\Gamma}$ of K (by sets U_α open in X) there exist fintely many indices $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Gamma$ for which

$$\bigcup_{j=1}^{k} U_{\alpha_j} \supset K.$$

That is to say, any open cover $\{U_\alpha\}_{\alpha\in\Gamma}$ of K admits a **finite subcover**.

The requirement that the reduction to a finte subcover is possible for **any** open cover is crucial. Some open covers of $\mathbb{R}^n$ admit finite subcovers, but $\mathbb{R}^n$ is certainly not compact.

Exercise 13.20. Show $\mathbb{R}^n$ is not compact.

Exercise 13.21. Let $I = [a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$. Show that if A is a closed subset of $\mathbb{R}$ and $A \subset I = [a, b]$, then A is compact.

It is not true that every compact subset in a topological space is closed, but it is often true.

Exercise 13.22. Show any compact set K in a Hausdorff topological space X is closed.

Exercise 13.23. Show the continuous image of a compact set is compact. That is, if X and Y are topological spaces, $f \in C^0(X \to Y)$, and K is a compact subset of X, then

$$f(K) = \{f(x) : x \in K\}$$

is compact.

Quotient topology

A/the quotient topology is used in the definition of a piecewise affine embedded surface in Chapter C.

Other exercises every individual should do at least once

Exercise 13.24. Every closed subset of a compact set in a topological space is compact.

Exercise 13.25. Every open interval $I = (a, b)$ with $a, b \in \mathbb{R}$ and $a < b$ is homeomorphic to $\mathbb{R}$.

Exercise 13.26. Every open ball

$$B_r(\mathbf{p}) = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}$$

where $r > 0$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, and

$$|\mathbf{x} - \mathbf{p}| = \sqrt{\sum_{j=1}^{n} (x_j - p_j)^2}$$

is homeomorphic to $\mathbb{R}^n$.

13.2 Algebraic groups

An introduction to groups is also given in Chapter 16. Here I will restrict attention to an idiosyncratic definition immediately relevant for the following sections.

A set G with a function $+ : G \times G \to G$ (called an operation of addition with value denoted by $a + b$ for $(a, b) \in G \times G$) is an **additive group** if

(i) $(a + b) + c = a + (b + c)$ for all $a, b, c \in G$.

(ii) There is some $0 \in G$ for which $a + 0 = 0 + a = a$ for all $a \in G$.

(iii) For each $a \in G$, there exists an element $-a \in G$ for which $a + (-a) = (-a) + a = 0$.

If your set satisfies the first two conditions, having an associative operation of addition and an additive inverse, then you've got yourself an **additive monoid**.

13.3 Linear spaces

A **linear space**[4] $\mathcal{L}$ is an additive group with a function $\sigma : \mathbb{R} \times \mathcal{L} \to \mathcal{L}$ called an **operation of scaling** with values denoted $\sigma(\alpha, \ell) = \alpha\ell$ satisfying

(i) $\alpha(\beta\ell) = (\alpha\beta)\ell$ for $\alpha, \beta \in \mathbb{R}$ and $\ell \in \mathcal{L}$.

(ii) $(\alpha + \beta)\ell = \alpha\ell + \beta\ell$ for $\alpha, \beta \in \mathbb{R}$ and $\ell \in \mathcal{L}$.

(iii) $\alpha(\ell_1 + \ell_2) = \alpha\ell_1 + \alpha\ell_2$ for $\alpha \in \mathbb{R}$ and $\ell_1, \ell_2 \in \mathcal{L}$.

(iv) $1\ell = \ell$ for $\ell \in \mathcal{L}$.

The additive identity in a linear space is often denoted by $\mathbf{0}$ to distinguish this element from the additive identity $0 \in \mathbb{R}$. Of course, the element 1 appearing in **(iv)** is the multiplicative identity in $\mathbb{R}$; that is about the only thing that would make sense.

I'm going to suggest the elements in a linear space be called **filaments** or **linear filaments** in order to distinguish the elements in a vector space which can be called **vectors** and have both direction and magnitude. The nonzero filaments in a linear space can be said to have direction determined by the equivalence relation

$$\ell_1 \sim \ell_2 \qquad \Longleftrightarrow \qquad \ell_1 = \alpha\ell_2 \qquad \text{for some } \alpha > 0$$

with the direction being the associated equivalence class. There may be no natural notion of magnitude (or unit magnitude in particular) for filaments in a linear space.

13.4 Distance spaces (a.k.a. metric spaces)

A set X is a **distance space** if there is a function $d : X \times X \to [0, \infty)$ for which the following conditions hold:

(i) $d(x, y) = d(y, x)$ for $x, y \in X$.

(ii) $d(x, y) = 0$ if and only if $x = y$.

[4]Linear spaces are often called vector spaces, but we will reserve that terminology for a linear space with a norm defined on it. See section 13.5 below.

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for $x, y, z \in X$.

The function $d : X \times X \to [0, \infty)$ is called a **distance function** or a **metric distance function** or sometimes a metric, but we will try to avoid this last terminology. The condition **(i)** is said to express the fact that the distance is symmetric. The condition **(ii)** defines what it means for the distance to be **positive definite**, and condition **(iii)** is called the **triangle inequality**. These terms can and do have different meanings in different contexts.

Note carefully that a distance space is not required to be a linear space. Sometimes a distance space is (also) a linear space however. Every distance space is a topological space with a **topology induced by the distance function**. This topology is defined as follows: A set $U \subset X$ is open if and only if for every $p \in X$, there is some $r > 0$ for which

$$B_r(p) = \{x \in X : d(x, p) < r\} \subset U.$$

Exercise 13.27. Show the topology induced by a distance function satisfies the axioms required for a topology.

13.5 Normed spaces

A linear space V is a **normed space** or a **vector space** if there is a function $\| \cdot \| : V \times V \to [0, \infty)$ for which the following condtions hold:

(i) $\|v\| = 0$ if and only if $v = \mathbf{0}$ is the additive identity in the linear space.

(ii) $\|\alpha v\| = |\alpha| \; \|v\|$ for $\alpha \in \mathbb{R}$ and $v \in V$.

(iii) $\|v + w\| \leq \|v\| + \|w\|$ for $v, w \in V$.

The function $\| \cdot \| : X \times X \to [0, \infty)$ is called a **norm**. The condition **(i)** is said to express the fact that the norm is **positive definite**. The condition **(ii)** defines what it means for the norm to be **nonnegative homogeneous**, and condition **(iii)** is called the **triangle inequality** for the norm.

Note carefully that a normed space **is** required to be a linear space. Every normed space is also a distance space with a **norm induced distance** given by $d : V \times V \to [0, \infty)$ given by

$$d(v, w) = \|w - v\|. \tag{13.3}$$

Exercise 13.28. Show the norm induced distance defined by (13.3) satisfies the axioms required to make V a distance space.

13.6 Inner product spaces

A linear space V is an **inner product space** if there is a function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ for which the following condtions hold:

(i) $\langle v, w\rangle = \langle w, v\rangle$ for $v, w \in V$.

(ii) $\langle \alpha v + \beta w, z\rangle = \alpha\langle v, z\rangle + \beta\langle w, z\rangle$ for $\alpha, \beta \in \mathbb{R}$ and $v, w, z \in V$.

(iii) $\langle v, v\rangle \geq 0$ with equality if and only if $v = \mathbf{0}$ is the additive identity in the linear space.

The function $\langle\ ,\ \rangle : X \times X \to \mathbb{R}$ is called an **inner product**. The condition **(i)** is said to express the fact that the norm is symmetric. The condition **(ii)** defines what it means for the norm to be **linear** in the first slot. By symmetry, the inner product is linear in the second slot as well and is thus referred to as **bilinear**. Condition **(iii)** expresses the condition that the inner product is **positive definite**.

Like a normed space, an inner product space is required to be a linear space, and in fact every inner product space is also a normed/vector space (and a distance space) with an **inner product induced norm** given by $\|\ \cdot\ \| : V \times V \to [0, \infty)$ given by

$$\|v\| = \sqrt{\langle v, v\rangle} \tag{13.4}$$

and distance given by

$$d(v, w) = \sqrt{\langle v - w, v - w\rangle}.$$

Exercise 13.29. Show the inner product induced norm defined by (13.4) satisfies the axioms required to make V a normed space.

13.7 Spaces of differentiable functions