1.5 Two steps backward

1.6 Cayley: global linearity

By Cayley geometry I mean essentially the Riemannian geometry associated with (or induced by) a linear or affine mapping of $\mathbb{R}^n$. In this section I will let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$\psi(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

where $A \in GL_n(\mathbb{R})$ is an $n \times n$ invertible matrix (with constant entries) and $\mathbf{b}$ is a fixed affine translation vector. We will denote the inverse of ψ by $\phi = \psi^{-1}$, the matrix A by (a_{ij}), and the inverse A^{-1} by (a^{ij}) where the entries in the inverse matrix are indexed by superscripts.

In principle we may consider the Riemannian manifold $\mathbb{R}^n_{(g_{ij})}$ with global chart $\mathbb{R}^n$ equipped with any positive definite symmetric matrix assignment (g_{ij}) and broaden the notion of Cayley geometry to linear changes of variable. As a start (at least when we do any semi-serious calculation) let us begin simply with flat Euclidean space $\mathbb{R}^n$, and for the moment we will focus on the case $n = 2$ (the plane) and the case with $\mathbf{b} = \mathbf{0}$. See Figure 1.10.

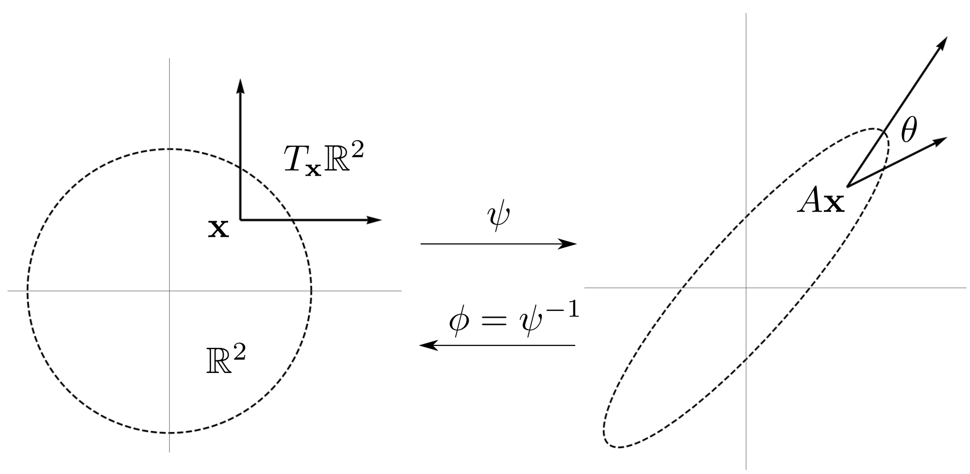

Figure 1.10: A nonsingular linear mapping of the plane

When $\mathbb{R}^n$, or $\mathbb{R}^2$ in this case, is considered as a Riemannian manifold we take the matrix assignment I associated with each point $\mathbf{x} \in \mathbb{R}$. As a result, the Euclidean length of a C^1 path $\alpha : [a,b] \to \mathbb{R}^n$ is given by

$$\text{length}[\alpha] = \int_{(a,b)} \sqrt{\langle I\alpha', \alpha' \rangle}$$

as usual.

Exercise 1.46. Find the matrix assignment (g_{ij}) for which the path $\beta : [a,b] \to \mathbb{R}^2$ given by $\beta = \psi \circ \alpha$ has length satisfying

$$\text{length}_{(g_{ij})}[\beta] = \int_{(a,b)} \sqrt{\langle (g_{ij})\alpha', \alpha' \rangle} = \text{length}[\alpha]$$

for every C^1 path $\alpha : [a,b] \to \mathbb{R}^2$.

Exercise 1.47. Given a constant $n \times n$ matrix (g_{ij}) which is positive definite and symmetric, does there exist a matrix $(a_{ij}) \in GL_n(\mathbb{R})$ with

$$(a^{ij})^T (a^{ij}) = (g_{ij})? \tag{1.32}$$

Exercise 1.48. Given a constant $n \times n$ matrix (g_{ij}) which is positive definite and symmetric, find all matrices (g^{ij}) for which (1.32) holds. Hint: Consider $(g_{ij}) = (\delta_{ij}) = I$ and remember the orthogonal matrices $O_n(\mathbb{R})$.

Exercise 1.49. Given

$$A = \frac{1}{4} \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix},$$

complete the following:

(a) Find the matrix assignment (g_{ij}) described in Exercise 1.46.

(b) Find all matrices for which (1.32) holds as in Exercises 1.47 and 1.48.

1.7 Gauss: surfaces

1.8 Riemann: charts