mined by

$$\begin{aligned}
&0+0=0\\
&0+1=1+0=0\\
&0+2=2+0=0\\
&0+3=3+0=0\\
&1+1=1\\
&1+2=2+1=2\\
&1+3=3+1=3\\
&2+2=0\\
&2+3=3+2=2\\
&3+3=1.
\end{aligned}$$

(i) Show the operation is associative and there is an identity element, so A is a monoid under this operation.

(ii) Show that $\mathbb{Z}_4$ is not a group under this operation.

(iii) Find all subsets of $\mathbb{Z}_4$ which are groups under this opearation.

(iv) How does what you found in part **(iii)** harmonize with Exercise 16.5?

(v) If we define a **subgroup** H of a group G to be a subset of G which is itself a group (under the same operation), determine which of the (distinct) groups you found in part **(iii)** are subgroups one of another.

Note: The Wikipedia page on "Green's relations" has a nice discussion and example related to this exercise.

16.2 Matrix groups

The $n \times n$ matrices with real entries, mentioned above and denoted by $\mathbb{R}^{n\times n}$, do not constitute a group under matrix multiplication, though they do constitute a group under addition. In fact, one proper algebraic classification for $\mathbb{R}^{n\times n}$ is that $\mathbb{R}^{n\times n}$ is a **ring**.

Definition 21. (ring) A set R is called a **ring** if there are operations of addition and multiplication for which the set R and the operations together satisfy the following conditions:

(i) For every $a, b, c \in R$, there holds

$$(a + b) + c = a + (b + c). \tag{16.3}$$

(ii) There is a zero element $0 \in R$ for which

$$a + 0 = 0 + a = a \tag{16.4}$$

for every $a \in R$.

(iii) For each $a \in R$, there is an additive inverse $-a \in R$ for which

$$a + (-a) = (-a) + a = 0.$$

The conditions **(i)**-**(iii)** say that R is a group under addition.

(iv) For each $a, b \in R$ there holds

$$a + b = b + a.$$

That is, R as a group under addition is a commutative or Abelian group.

(v) For every $a, b, c \in R$, there holds

$$(ab)c = a(bc). \tag{16.5}$$

(vi) There is a multiplicative identity $1 \in R$ for which

$$a1 = 1a = a \tag{16.6}$$

for every $a \in R$.

It may be noted that conditions **(v)**-**(vi)** make R a monoid under multiplication. Note that the existence of inverses under multiplication is not required and is not true in $\mathbb{R}^{n\times n}$ where the multiplicative identity is the identity matrix. Also, commutativity of the multiplication is not required and does not hold in $\mathbb{R}^{n\times n}$ for $n > 1$. Note finally, that property **(vi)** is not expressing anything about any kind of scaling. In particular, if this definition is applied to the $n \times n$ matrices, then the symbol 1 should be interpreted as the identity matrix $1 \in \mathbb{R}^{n\times n}$ rather than the scalar $1 \in \mathbb{R}$, though in that case the scaling $1a$ or more generally ca for $c \in \mathbb{R}$ and $a \in \mathbb{R}^{n\times n}$ does make sense and appears notationally identical.

(vii) Multiplication is distributive across addition: For $a, b, c \in R$ there holds

$$a(b+c) = ab + ac \tag{16.7}$$
$$(a+b)c = ac + bc. \tag{16.8}$$

Note that the ring multiplication is not required to be commutative here, so there are left distributive (16.7) and a right distributive (16.8) properties. Again, the identities (16.7) and (16.8) have nothing to do with scaling as encountered with a vector space or a module.

When the algebraic structure of a ring is applied to $\mathbb{R}^{n\times n}$ the emphasis is usually on the operation of multiplication of matrices. This may be said to hold more generally as well. A ring is said to be a **commutative ring** if the multiplication is commutative. An element in a ring for which a multiplicative inverse exists is called **invertible**.

Exercise 16.10. Verify the following assertions concerning a ring R and the ring $\mathbb{R}^{n\times n}$ of $n \times n$ matrices in particular:

(a) $0a = 0$ for every $a \in R$.

(b) If $a \in R$ and there is some $b \in R$ for which $ab = 1$, then the element b is called a **right inverse** for a. Show that if a_1 and a_2 have right inverses, then $a_1 a_2$ has a right inverse.

(c) Show

$$R^\times = \{a \in R : \text{for some } b, c \in R \text{ there holds } ab = ca = 1\}$$

is closed under multiplication.

(d) If $a \in R^\times$ and b denotes a right inverse of a and c denotes a left inverse of a, then $b = c$. Hint: Remember the proof that inverses in a group are unique. In particular, if $ab = ba = 1$ and $a\tilde{b} = \tilde{b}a = 1$, then $b = b(1) = b(a\tilde{b})$.

(e) Conclude from part **(d)** that

$$R^\times = \{a \in R : \text{for some } b \in R \text{ there holds } ab = ba = 1\},$$

and show $R^\times$ is a multiplicative group. Hint: It still remains to show that if $ba = 1$ for some $a \in R^\times$, then $b \in R^\times$.

(f) The multiplicative inverses in a ring are unique.

The multiplicative group $R^\times$ considered above is called the group of **invertible elements**.

Exercise 16.11. Identify $R^\times$ for the ring $R = \mathbb{R}^{n\times n}$.

Exercise 16.12. Verify the following assertions concerning a ring R and the ring $\mathbb{R}^{n\times n}$ of $n \times n$ matrices in particular:

(a) $Z(R) = \{a \in R : ab = ba$ for every $b \in R\}$ is a subring of R. This ring is called the **center** of the ring R or the center of the monoid under multiplication.

(b) $R^\times \cap Z(R)$ is a group (under multiplication).

(c) If $R^\times \cap Z(R) = Z(R)$, then $R = \{0\}$.

Exercise 16.13. Identify $Z(R)$ and the subgroup $R^\times \cap Z(R)$ for the ring $R = \mathbb{R}^{n\times n}$.

Exercise 16.14. In a general ring R if $0 = 1$, then $R = \{0\}$.

Exercise 16.15. Give an example of a pair of 2×2 matrices in $\mathbb{R}^{2\times 2}$ which do not commute under matrix multiplication.

16.3 Warm Up

The following exercises give a suggestion for a direction in understanding something about Lie groups.

Exercise 16.16. Describe the following subsets of $\mathbb{R}^3$:

(a) $\{\mathbf{x} = (x_1, x_2, x_3) : x_1x_3 - x_2 = 0\}$.

(b) $R = \{\mathbf{x} = (x_1, x_2, x_3) : x_1x_3 - x_2 < 0\}$.

(c) $L = \{\mathbf{x} = (x_1, x_2, x_3) : x_1x_3 - x_2 > 0\}$.

Exercise 16.17. Describe the following subsets of $\mathbb{R}^2$:

(a) $\{\mathbf{x} = (x_2, x_3) : (x_1, x_2, x_3) \in \partial R,\ x_1 < 0\}$.

(b) $\{\mathbf{x} = (x_2, x_3) : (x_1, x_2, x_3) \in \partial R,\ x_1 = 0\}$.

(c) $\{\mathbf{x} = (x_2, x_3) : (x_1, x_2, x_3) \in \partial R,\ x_1 > 0\}$.

Exercise 16.18. Given the set $\Omega \subset \mathbb{R}^2$ (described in each part below) find an extended real valued function $h : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ for which

$$\begin{aligned}&\{\mathbf{x} = (x_1, x_2, x_3) \in R : (x_1, x_2) \in \Omega\}\\ &\qquad\qquad = \{\mathbf{x} = (x_1, x_2, x_3) : (x_1, x_2) \in \Omega, x_3 > h(x_1, x_2)\}.\end{aligned}$$

A set of this form is called a **supergraph**.

(a) $\Omega = \Omega_- = \{\mathbf{x} = (x_1, x_2) : x_1 < 0\}$.

(b) $\Omega = \Omega_0 = \{\mathbf{x} = (x_1, x_2) : x_1 = 0\}$.

(c) $\Omega = \Omega_+ = \{\mathbf{x} = (x_1, x_2) : x_1 > 0\}$.

Exercise 16.19. Let $\Omega = \mathbb{R}^n$ and consider $h :\in C^0(\Omega)$. Show the following:

(a) $A = \{(\mathbf{x}, x_{n+1}) : \mathbf{x} = (x_1, x_2, \dots, x_n),\ x_{n+1} < h(\mathbf{x})\}$ is an open connected subset of $\mathbb{R}^{n+1}$. In particular, the identity is a global chart function making A a manifold. (For example, a smooth Riemannian manifold with matrix assignment the constant identity matrix.)

(b) $B = \{(\mathbf{x}, x_{n+1}) : \mathbf{x} = (x_1, x_2, \dots, x_n),\ x_{n+1} > h(\mathbf{x})\}$ is an open connected subset of $\mathbb{R}^{n+1}$.

(c) $\partial A = \partial B = \mathcal{G}$ where

$$\mathcal{G} = \{(\mathbf{x}, x_{n+1}) : \mathbf{x} = (x_1, x_2, \dots, x_n),\ x_{n+1} = h(\mathbf{x})\}.$$

Exercise 16.20. Let $\mathbb{R}^{n\times n}$ denote the set of $n \times n$ matrices $a = (a_{ij})$ with real entries. Show the following:

(a) $\mathbb{R}^{n\times n}$ is a group under (matrix) addition.

(b) $\mathbb{R}^{n\times n}$ is a vector space over $\mathbb{R}$.

(c) $\mathbb{R}^{n\times n}$ is a monoid under matrix multiplication.

(d) $\mathbb{R}^{n\times n}$ is a ring under matrix addition and multiplication.

Exercise 16.21. (column bijection) Consider

$$\psi_c : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$$

by

$$\psi_c(a) = (c_1^T, c_2^T, \ldots, c_n^T)$$

where

$$a = (a_{ij}) = (c_1\ c_2\ \ldots\ c_n) \qquad \text{and} \qquad c_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

denotes the j-th column of $a = (a_{ij})$ for $j = 1, 2, \ldots, n$.

(a) Show ψ_c is a group isomorphism of additive groups.

(b) Show ψ_c is a vector space isomorphism of real vector spaces.

(c) Show ψ_c is a monoid isomorphism with respect to multiplication. (Write down the multiplication formula for $\mathbf{xy}$ where $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $\mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. Can you generalize this to $\mathbb{R}^{n^2}$?)

(d) Show ψ_c is a ring isomorphism.

Exercise 16.22. (row bijection) Consider

$$\psi_r : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$$

by

$$\psi_r(a) = (r_1, r_2, \ldots, r_n)$$

where

$$a = (a_{ij}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \qquad \text{and} \qquad r_i = (a_{i1}, a_{i2}, \cdots, a_{in})$$

denotes the i-th row of $a = (a_{ij})$ for $i = 1, 2, \ldots, n$.

(a) Show ψ_r is a group isomorphism of additive groups.

(b) Show ψ_r is a vector space isomorphism of real vector spaces.

(c) Show ψ_r is a monoid isomorphism with respect to multiplication. (Write down the multiplication formula for $\mathbf{xy}$ where $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $\mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. Can you generalize this to $\mathbb{R}^{n^2}$?)

(d) Show ψ_r is a ring isomorphism.

Exercise 16.23. ($GL_n(\mathbb{R})$) The group of invertible elements $R^\times$ considered in Exercise 16.10 when $R = \mathbb{R}^{n\times n}$ is the ring of real matrices is called $GL_n(\mathbb{R})$. In this case, we have a characterization of the invertible matrices in terms of a real valued function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ called the determinant.

(a) Use one of the bijections ψ_c or ψ_r to induce a topology on $\mathbb{R}^{n\times n}$ and Show det is continuous.

(b) Conclude $\psi_r(GL_n(\mathbb{R}))$ and $\psi_c(GL_n(\mathbb{R}))$ are open subsets of $\mathbb{R}^{n^2}$.

(c) What is the relation between $\psi_r(GL_n(\mathbb{R}))$ and $\psi_c(GL_n(\mathbb{R}))$?

(d) Show $GL_n^+(\mathbb{R}) = \{a \in \mathbb{R}^{n\times n} : \det(a) > 0\}$ is a group under (matrix) multiplication.

(e) Is $\mathbb{R}^{n\times n}\backslash GL_n(\mathbb{R})$ closed under matrix multiplication?

Exercise 16.24. ($GL_1(\mathbb{R})$)

(a) What is $\mathbb{R}^{1\times 1}\backslash GL_1(\mathbb{R})$?

(b) What is $GL_1^+(\mathbb{R})$?

(c) What is $GL_1(\mathbb{R})$?

(d) $GL_1^+(\mathbb{R})$ and $GL_1(\mathbb{R})\backslash GL_1^+(\mathbb{R})$ are homeomorphic one-dimensional manifolds each of which admits a global chart. Give global charts $\mathbf{p}_+$ for $GL_1^+(\mathbb{R})$ and $\mathbf{p}_-$ for $GL_1(\mathbb{R})\backslash GL_1^+(\mathbb{R})$, and give a homeomorphism $\psi : GL_1^+(\mathbb{R}) \to GL_1(\mathbb{R})\backslash GL_1^+(\mathbb{R})$.

(e) Which of the following manifolds is a Lie group:

1. $GL_1^+(\mathbb{R})$.
2. $GL_1(\mathbb{R})\backslash GL_1^+(\mathbb{R})$.
3. $GL_1(\mathbb{R})$.
4. $\mathbb{R}^{1\times 1}\backslash GL_1(\mathbb{R})$.

(f) $\mathbb{R}^{n\times n}\backslash GL_n(\mathbb{R})$ is not a Lie group for $n > 1$. Why not?

Exercise 16.25. $(GL_2(\mathbb{R}))$ Express the following as sets in $\mathbb{R}^4$:

(a) $\psi_c(GL_2(\mathbb{R}))$.

(b) $\psi_c(GL_2^+(\mathbb{R}))$.

(c) $\psi_c(\mathbb{R}^{2\times 2}\backslash GL_2(\mathbb{R}))$.

(d) $\psi_r(GL_2(\mathbb{R}))$.

(e) $\psi_r(GL_2^+(\mathbb{R}))$.

(f) $\psi_r(\mathbb{R}^{2\times 2}\backslash GL_2(\mathbb{R}))$.

Example: **(a)** $\psi_c(GL_2(\mathbb{R})) = \{\mathbf{x} = (x_1, x_2, x_3, x_4) : x_1x_4 - x_2x_3 \neq 0\}$. The next exercises are intended to give further insight in to the nature of these particular sets.

Exercise 16.26. You may wish to review Exercises 16.16-16.19 before undertaking this exercise.

(a) Let $\Omega_- = \{(\mathbf{x} = (x_1, x_2, x_3) : x_1 < 0\}$. Find a function $h \in C^0(\Omega_-)$ for which

$$\begin{aligned}\{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \psi_r(GL_2^+(\mathbb{R})) : (x_1, x_2, x_3) \in \Omega_-\}\\ = \{\mathbf{x} = (x_1, x_2, x_3, x_4) : (x_1, x_2, x_3) \in \Omega_-,\ x_4 > h(x_1, x_2, x_3)\}\end{aligned}$$

and

$$\begin{aligned}\{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \psi_r(\mathbb{R}^{2\times 2}\backslash GL_2(\mathbb{R})) : (x_1, x_2, x_3) \in \Omega_-\}\\ = \{\mathbf{x} = (x_1, x_2, x_3, x_4) : (x_1, x_2, x_3) \in \Omega_-,\ x_4 = h(x_1, x_2, x_3)\}.\end{aligned}$$

(b) Let $\Omega_+ = \{(\mathbf{x} = (x_1, x_2, x_3) : x_1 > 0\}$. Find a function $h \in C^0(\Omega_+)$ for which

$$\begin{aligned}\{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \psi_r(GL_2^+(\mathbb{R})) : (x_1, x_2, x_3) \in \Omega_+\} \\ = \{\mathbf{x} = (x_1, x_2, x_3, x_4) : (x_1, x_2, x_3) \in \Omega_+,\ x_4 > h(x_1, x_2, x_3)\}\end{aligned}$$

and

$$\begin{aligned}\{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \psi_r(\mathbb{R}^{2\times 2}\backslash GL_2(\mathbb{R})) : (x_1, x_2, x_3) \in \Omega_+\} \\ = \{\mathbf{x} = (x_1, x_2, x_3, x_4) : (x_1, x_2, x_3) \in \Omega_+,\ x_4 = h(x_1, x_2, x_3)\}.\end{aligned}$$

(c) Let $\Omega_0 = \{(\mathbf{x} = (x_1, x_2, x_3) : x_1 = 0\}$. Describe/make an illustration of

$$\{\mathbf{x} = (x_1, x_2, x_3) \in \Omega_0 : (x_1, x_2, x_3, x_4) \in \partial\psi_k(GL_2(\mathbb{R}))\}.$$

Hint: Make first an illustration of

$$\{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \partial\psi_k(GL_2(\mathbb{R})) : (x_1, x_2, x_3) \in \Omega_0\}.$$

Exercise 16.27. How many components does $\psi_r(GL_2^+(\mathbb{R}))$ have and what are they? (Fully justify your answer.)

Exercise 16.28. Give a global chart for the manifold

$$\psi_r(\mathbb{R}^{2\times 2}\backslash GL_2(\mathbb{R})) \bigcap \{\mathbf{x} = (x_1, x_2, x_3, x_4) : (x_1, x_2, x_3) \in \Omega_-\}$$

where Ω_- is defined in Exercise 16.27 part **(a)**.

Exercise 16.29. Describe the set

$$\Sigma = \psi_r(\mathbb{R}^{2\times 2}\backslash GL_2(\mathbb{R})) = \partial\psi_2(GL_2(\mathbb{R})).$$

Note $\psi_r(GL_2(\mathbb{R})) = \mathbb{R}^4\backslash\Sigma$.

16.4 The matrix ring $\mathbb{R}^{2\times 2}$

As mentioned above $\psi : \mathbb{R}^{2\times 2} \to \mathbb{R}^4$ by $\psi(a_{ij}) = (a_{11}, a_{21}, a_{12}, a_{22})$ is a bijection, and by means of this bijection we can induce simple topological, vector space, normed space, inner product space, and Riemannian manifold structures on $\mathbb{R}^{2\times 2}$. In turn, the ring structure involving the matrix multiplication in $\mathbb{R}^{2\times 2}$ may be induced on $\mathbb{R}^4$. Of course, the flat Riemannian manifold $\mathbb{R}^4$ is not a group with respect to the (induced matrix) multiplcation, but it is a ring. Various subsets of $\mathbb{R}^{2\times 2}$ and by the bijective correspondence $\mathbb{R}^4$ may be considered. In particular, $GL_2(\mathbb{R})$ is a group, and it may be interesting to understand the corresponding subset of $\mathbb{R}^4$. Within $GL_2(\mathbb{R})$ are the **rotation matrices**

$$SL_2(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) : \det A = 1\}$$

and the larger set of **orthogonal matrices**

$$O_2(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) : |\det A| = 1\}.$$

These are also both (sub)groups under matrix multiplication. Other subsets which are not groups (or subgroups) are the collection of **symmetric matrices**

$$\mathrm{Sym}_n(\mathbb{R}) = \{A \in \mathbb{R}^{2\times 2} : A^T = A\}$$

and the collection

$$O_2(\mathbb{R})\backslash SL_2(\mathbb{R}).$$

It may be interesting to consider for each of these sets the bijectively corresponding subsets, and possibly submanifolds, of $\mathbb{R}^4$. Finally, consideration of $GL_2(\mathbb{R})$ naturally suggests the consideration of $\mathbb{R}^{2\times 2}\backslash GL_2(\mathbb{R})$ and the bijectively corresponding set in $\mathbb{R}^4$.

Exercise 16.30. Show

$$O_2(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : A^T = A^{-1}\}.$$

16.5 $\mathbb{S}^1$ as a group

I haven't given a formal definition of a Lie group yet, but that should not stop us from considering an example of one. We know $\mathbb{S}^1$ is a C^∞ submanifold of $\mathbb{R}^2$. **Universal covering maps** of $\mathbb{S}^1$ are given by $\mathbf{p} : \mathbb{R} \to \mathbb{S}^1$ and $\mathbf{q} : \mathbb{R} \to \mathbb{S}^1$ by

$$\mathbf{p}(t) = (\cos t, \sin t) \qquad \text{and} \qquad \mathbf{q}(t) = (\cos(2\pi t), \sin(2\pi t)) \tag{16.9}$$

respectively. If $P_1, P_2 \in \mathbb{S}^1$ and $t_j \in \mathbb{R}$ with $\mathbf{p}(t_j) = P_j$ for $j = 1, 2$, then $\mathbf{p}(t_1 + t_2)$ determines a unique point in $\mathbb{S}^1$. To see this, note that for $j = 1, 2$ we have

$$\mathbf{p}(P_j) = \{t \in \mathbb{R} : (\cos t, \sin t) = P_j\} = \{t_j + 2\pi k : k \in \mathbb{Z}\}.$$

Thus, for each $\tilde{t}_j \in \mathbf{p}^{-1}(P_j)$, $j = 1, 2$ there is some k_j with $\tilde{t}_j = t_j + 2\pi k_j$. Therefore,

$$\begin{aligned}\mathbf{p}(\tilde{t}_1 + \tilde{t}_2) &= (\cos[t_1 + t_2 + 2\pi(k_1 + k_2)], \sin[t_1 + t_2 + 2\pi(k_1 + k_2)]\\ &= (\cos(t_1 + t_2), \sin(t_1 + t_2))\\ &= \mathbf{p}(t_1 + t_2).\end{aligned}$$

. Thus, setting $P_1 + P_2 = \mathbf{p}(t_1 + t_2)$ gives a well-defined operation of addition on $\mathbb{S}^1 \times \mathbb{S}^1$.

Exercise 16.31. Show the operation $+ : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ defined by $P_1 + P_2 = \mathbf{p}(t_1 + t_2)$ above is associative and makes $\mathbb{S}^1$ a commutative group under addition.

Exercise 16.32. Illustrate with a drawing the group operation $+ : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ considered in Exercise 16.31

Exercise 16.33. Show that for each $P_0 \in \mathbb{S}^1$, the function $f : \mathbb{S}^1 \to \mathbb{S}^1$ by $f(P) = P + P_0$ satisfies $f \in C^\infty(\mathbb{S}^1 \to \mathbb{S}^1)$.

Exercise 16.34. Can you make sense of what it would mean for the function $f \in C^\infty(\mathbb{S}^1 \to \mathbb{S}^1)$ from the previous problem to satisfy $f \in C^\omega(\mathbb{S}^1 \to \mathbb{S}^1)$.

Exercise 16.35. Use the universal covering map $\mathbf{q} : \mathbb{R} \to \mathbb{S}^1$ defined in (16.9) above to define a group structure on $\mathbb{S}^1$. Do you get the same group addition or a different one?

16.6 $SL_2(\mathbb{R})$

Exercise 16.36. Use the (restriction of the) canonical bijection $\psi : \mathbb{R}^{2\times 2} \to \mathbb{R}^4$ to obtain a universal covering map $\mathbf{p} : \mathbb{R} \to \psi(SL_2(\mathbb{R})$. Show the image $\mathbf{p}(\mathbb{R})$ is a circle (in a flat two-plane) in

$$\mathbb{S}^3_{\sqrt{2}} = \{\mathbf{x} = (x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2\}.$$

Exercise 16.37. Use (a) stereographic projection $\sigma : \mathbb{S}^3_{\sqrt{2}} \to \mathbb{R}^3$ to visualize $\psi(SL_2(\mathbb{R}))$ and $\psi(O_2(\mathbb{R})\backslash SL_2(\mathbb{R}))$ in $\mathbb{R}^4$.

Exercise 16.38. Use your visualization from Exercise 16.37 to illustrate with a drawing the group operation of matrix multiplication in $\mathbb{R}^{2\times 2}$ on $SL_2(\mathbb{R})$ induced on $\psi(SL_2(\mathbb{R})) \subset \mathbb{R}^4$.

Exercise 16.39. Group action of $SL_2(\mathbb{R})$ on $O_2(\mathbb{R})$.