Some comments about the trace in H_0^1

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Let us imagine that we wish to consider the infemum of the set

$$\{M : (u^+ - M)^+ \in H_0^1(\mathcal{U})\}$$

for some function $u \in H_0^1(\mathcal{U})$ defined on an open bounded set $\mathcal{U}$. It might be convenient to note that only nonnegative values of M need be considered. That is,

$$\{M : (u^+ - M)^+ \in H_0^1(\mathcal{U})\} = \{M \geq 0 : (u^+ - M)^+ \in H_0^1(\mathcal{U})\}.$$

Notice that for $M < 0$, we have $(u^+ - M)^+ = u^+ - M \geq -M > 0$. Therefore, the equality of the two sets follows from the following interesting result.

Theorem 1 *If $u \in H^1(\mathcal{U})$ satisfies $u \geq \epsilon > 0$, then $u \notin H_0^1(\mathcal{U})$.*

This result is much more difficult than one might guess, and it is our objective here to outline the proof through a series of lemmas. Some of these will require a slightly more general setting and general arguments than we have considered before.

For this discussion, let $\mathcal{U}$ denote an open bounded domain in $\mathbb{R}^n$ and Ω simply an open domain.

Let us denote by $W^1(\Omega)$ the collection of all functions in $L^1_{\text{loc}}(\Omega)$ with weak first partial derivatives in $L^1_{\text{loc}}(\Omega)$.

We begin with a relatively easy result from Gilbarg and Trudinger. (Though the proof given there seems to have a minor technical error.)

Lemma 1 *If $u \in W^1(\Omega)$, then $u^+(x) = \max\{u(x), 0\}$ satisfies $u^+ \in W^1(\Omega)$ with*

$$Du^+(x) = \begin{cases} Du(x), & x \in \{\xi : u(\xi) > 0\} \\ 0, & x \in \{\xi : u(\xi) \leq 0\}. \end{cases}$$

Since $u^- = \min\{u(x), 0\} = -(u^-)^+$ and $|u| = u^+ - u^-$, we also obtain the related results:

Lemma 2 *If $u \in W^1(\Omega)$, then $u^- \in W^1(\Omega)$ with*

$$Du^-(x) = \begin{cases} Du(x), & x \in \{\xi : u(\xi) < 0\} \\ 0, & x \in \{\xi : u(\xi) \geq 0\}, \end{cases}$$

and $|u| \in W^1(\Omega)$ with

$$D|u|(x) = \begin{cases} Du(x), & x \in \{\xi : u(\xi) > 0\} \\ 0, & x \in \{\xi : u(\xi) = 0\} \\ -Du(x), & x \in \{\xi : u(\xi) < 0\}. \end{cases}$$

Using these results we find the following.

Lemma 3 *If u is weakly differentiable and is constant on a particular set A, then $Du = 0$ on A.*

Hints: Let the constant be c. Then $v = u - c$ vanishes on A. Since v is weakly differentiable, we have that $Dv = Dv^+ + Dv^-$.

Among other results, we will obtain a kind of converse for this result. The proof of the converse outlined below comes basically from Adams' book *Sobolev Spaces*.

Before getting to that, however, we note the following corollary.

Corollary 1 *If $u \in H_0^1(\mathcal{U})$, then for $\epsilon > 0$, the function $\underline{u}(x) = \min\{u(x), \epsilon\}$ satisfies $\underline{u} \in H_0^1$.*

Corollary 2 *If we assume $u \in H_0^1(\mathcal{U})$ and $u \geq \epsilon > 0$, then the constant function $\underline{u}(x) \equiv \epsilon$ is in $H_0^1(\mathcal{U})$.*

It is our main objective to show this second corollary is vacuous, i.e., there can be no such u.

Lemma 4 *(zero extension) If $u \in H_0^1(\mathcal{U})$, then*

$$\underline{u}(x) = \begin{cases} u(x), & x \in \mathcal{U} \\ 0, & x \in \mathbb{R}^n \backslash \mathcal{U} \end{cases}$$

satisfies $\underline{u} \in H_0^1(\mathbb{R}^n)$.

Putting these results together under the assumption that we have a function $u \in H_0^1(\mathcal{U})$ bounded from below by $\epsilon > 0$, we conclude that $v = \epsilon\chi_{\mathcal{U}} \in H_0^1(\mathbb{R}^n)$. Applying Lemma 3 to v, we know that $Dv \equiv 0$. The converse mentioned above is the following:

Theorem 2 *If $v \in H_0^1(\mathbb{R}^n)$ satisfies $Dv = 0$, then there is some constant c for which $v = c$ almost everywhere.*

The condition that the weak derivative vanishes here means

$$\int vD_j\phi = 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n) \text{ and } j = 1, \ldots, n.$$

In order to use this condition, we will use a generalization of the following result which characterizes the C_c^∞ functions which are also derivatives of C_c^∞ functions.

Lemma 5 *For $\phi \in C_c^\infty(\mathbb{R})$, the following are equivalent:*

(i) *$\phi = \psi'$ for some $\psi \in C_c^\infty(\mathbb{R})$.*

(ii) *$\int \phi = 0$.*

The generalization is the following:

Lemma 6 *(Adams' lemma) For $\phi \in C_c^\infty(\mathbb{R}^n)$, the following are equivalent:*

(i) *$\phi = \sum_{j=1}^n D_j\psi_j$ for some $\psi_1, \ldots, \psi_n \in C_c^\infty(\mathbb{R})$.*

(ii) *$\int \phi = 0$.*

Proof of Theorem 2: Consider $\ell : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ by

$$\ell[\phi] = \int u\phi.$$

If $\ell \equiv 0$, then the measure theoretic version of the fundamental lemma[1] implies $u = 0$ almost everywhere and we are done. Thus, we may assume there is some $\phi_0 \in C_c^\infty(\mathbb{R}^n)$ with $\ell[\phi_0] = c_0 \neq 0$.

Let us consider the value

$$c_1 = \int \phi_0.$$

[1]See the auxiliary results.

If $c_1 = 0$, then by Adams' lemma, we can write $\phi_0 = \sum D_j\psi_j$, and

$$\ell[\phi_0] = \sum \ell[D_j\psi_j] = \sum \int uD_j\psi_j = 0.$$

This contradicts our assumption that $c_0 \neq 0$. Therefore, we have $c_1 \neq 0$. This means we can compute for any $\phi \in C_c^\infty(\mathbb{R}^n)$

$$\int \left(\phi - \frac{\delta}{c_1}\phi_0\right) = \int \phi - \frac{\delta}{c_1}\int \phi_0$$

where

$$\delta = \int_{\mathbb{R}^n} \phi$$

is considered constant. Since $\int \phi_0 = c_1$, the calculation gives

$$\int \left(\phi - \frac{\delta}{c_1}\phi_0\right) = 0.$$

This means we can apply Adams' lemma again and write

$$\phi - \frac{1}{c_1}\left(\int \phi\right)\phi_0 = \sum D_j\psi_j$$

for some $\psi_1, \ldots, \psi_n \in C_c^\infty(\mathbb{R}^n)$. Computing in the same way we did above, we get

$$\ell\left(\phi - \frac{\delta}{c_1}\phi_0\right) = \sum \ell[D_j\psi_j] = \sum \int uD_j\psi_j = 0.$$

That is,

$$\ell[\phi] - \frac{\delta}{c_1}\ell[\phi_0] = 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n).$$

Rearranging the expression on the left and using the integral expression for δ, this means

$$\int \left(u - \frac{1}{c_1}\ell[\phi_0]\right)\phi = 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n).$$

Thus by the measure theoretic version of the fundamental lemma, we have $u = c = c_0/c_1$ is constant almost everywhere. □

Proof of Theorem 1: Continuing with the assumption above that $u \in H_0^1(\mathcal{U})$ with $u \geq \epsilon > 0$, we have observed that $v = \epsilon\chi_{\mathcal{U}} \in H_0^1(\mathbb{R}^n)$ and $Dv = 0$. According to Theorem 2, we must have v is constant almost everywhere, but we know $v = \epsilon$ on a set of positive measure, namely $\mathcal{U}$, and $v = 0$ on a set of positive measure, namely $\mathbb{R}^n\backslash\mathcal{U}$. Therefore, we have a contradiction, and the main result is established. □

Auxiliary results

Lemma 7 *(fundamental lemma) If* $u \in L^1_{\text{loc}}(\Omega)$ *and*

$$\int u\phi = 0 \quad \text{for all } \phi \in C_c^\infty(\Omega),$$

then $u = 0$ *almost everywhere.*