Math 6341, Final Exam Name and section: ____________________

1. (25 points) Solve the initial value problem

$$\begin{cases} (1+u^2)(u_t + u_x) = 1 \text{ on } \mathbb{R}\times(0,\infty) \\ u(x,0) = u_0(x). \end{cases}$$

Solution: Dividing both sides of the equation by the positive quantity $(1+u^2)$, the characteristic equation reads $\xi'(t) = (1,1)$. That is, $\xi(t) = x_0 + t$. Along the characteristics, we have

$$\frac{d}{dt}u(x_0+t,t) = \frac{1}{(1+u(x_0+t,t)^2)}.$$

That is,

$$\frac{1}{3}u(x_0+t,t)^3 + u(x_0+t,t) = \frac{1}{3}u_0(x_0)^3 + u_0(x_0) + t.$$

Replacing x_0 with $x-t$, we have

$$\frac{1}{3}u(x,t)^3 + u(x,t) = \frac{1}{3}u_0(x-t)^3 + u_0(x-t) + t.$$

Finally, noting that the function $\phi(f) = f^3/3 + f$ has $\phi' \geq 1$, this function has a well defined real smooth inverse. In fact,

$$\phi^{-1}(p) = \sqrt[3]{\frac{2}{\sqrt{4+9p^2}-3p}} - \sqrt[3]{\frac{\sqrt{4+9p^2}-3p}{2}}.$$

Thus,

$$u(x,t) = \sqrt[3]{\frac{2}{\sqrt{4+9[u_0(x-t)^3/3+u_0(x-t)+t]^2}-3[u_0(x-t)^3/3+u_0(x-t)+t]}} \\ - \sqrt[3]{\frac{\sqrt{4+9[u_0(x-t)^3/3+u_0(x-t)+t]^2}-3[u_0(x-t)^3/3+u_0(x-t)+t]}{2}}.$$

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2. (25 points) Consider the IVP

$$\begin{cases} u_t + uu_x = 0 \text{ on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = u_0 \end{cases}$$

where

$$u_0(x) = \begin{cases} 1 - x, & \text{if } |x| \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find two distinct integral solutions.

Find the unique entropy solution.

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3. (25 points) Let

$$u_0(x) = \begin{cases} 1 - |x|, & \text{if } |x| \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Consider the following variational problem: Minimize

$$\mathcal{F}[\mathbf{w}] = \int_a^b \left[\frac{\|\mathbf{w}'(t)\|^2}{2} - \alpha w_3(t) \right] dt + u_0(\mathbf{w}(a))$$

over the admissible class

$$\mathcal{A} = \left\{ \mathbf{w} = (w_1, w_2, w_3) \in C^2[a,b] : \mathbf{w}(b) = \mathbf{x}^* \right\}$$

where α is a positive constant.

(a) Find a formula $\mathbf{w}_0 = \mathbf{w}_0(t; \mathbf{x}^*, b)$ for the minimizer. Hint: Consider the minimization problem for $\mathcal{G}[\mathbf{w}] = \mathcal{F}[w] - u_0(\mathbf{w}(a))$ first.

(b) Assume $a = \alpha = 0$, and let $u(\mathbf{x}^*, t) = \mathcal{F}[\mathbf{w}_0]$ be the minimum value obtained above. Show that

$$u(x,t) = \min_{\xi \in \mathbb{R}^3} \left\{ \frac{|x - \xi|^2}{2t} + u_0(\xi) \right\}.$$

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(c) Find the Legendre transform L^* of $L(v) = \|v\|^2/2$.

(d) Set $H(p) = L^*(p)$ with L given above. Show that $u(x,t)$ is a weak solution of the IVP

$$\begin{cases} u_t + H(Du) = 0, & \text{on } \mathbb{R}^3 \times (0,\infty) \\ u(x,0) = u_0(x). \end{cases}$$

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Solution: This is a pretty tricky problem. Let's take it slowly and start with a task which is implicit in part (b). There is a calculus problem there which is pretty interesting on its own. That problem is:

> *Given* $(x,t) \in \mathbb{R}^3 \times (0,\infty)$ *fixed, minimize*
>
> $$f(\xi) = \frac{|x-\xi|^2}{2t} + u_0(\xi)$$
>
> *over* $\xi \in \mathbb{R}^3$.

To get the hang of how to do such problems, it is often a good idea to consider easier ones first, and an obvious notion of "easier" in this case is "lower dimensional." In accord with this strategy, let's minimize the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(\xi) = \frac{(\xi-x)^2}{2t} + u_0(\xi)$$

where we take u_0 with the same definiton but domain $\mathbb{R}$. Notice that

$$f(\xi) = \begin{cases} f_0(\xi) & \text{for } |\xi| \le 1 \\ f_1(\xi) & \text{for } |\xi| \ge 1 \end{cases}$$

where

$$f_0(\xi) = \frac{(\xi-x)^2}{2t} + 1 - |\xi|$$

and

$$f_1(\xi) = \frac{(\xi-x)^2}{2t}.$$

Since $1-|\xi| = 0$ when $|\xi| = 1$, it is clear that f is continuous. In fact, if we put $|\xi - x|^2$ back in place of $(\xi-x)^2$, these same observations would hold in the higher dimensional case.

Now, the function f_1 has a kind of strong uniform coercivity. To be precise, if $|\xi| \ge r = \max\{1, |x|\}$, we can be sure that $f(\xi)$ is increasing in $|\xi|$ and f_1, and hence f, attains a minimum somewhere on the closure of $B_r(0)$.

As a special case of this coercivity property, we note that when $|x| \le 1$, the minimum of f is always taken on the closure of $B_1(0)$ which is just $[-1,1]$ in this 1-D case.

We also observe that f_1 has a unique global min at $\xi = x$ with $f(x) = 0$ and is strictly convex everywhere. The other function f_0 is more complicated—and, in some sense, that is where the real action is happening. First of all, f_0 is differentiable for $\xi \neq 0$ with

$$f_0'(\xi) = \frac{\xi - x}{t} - \operatorname{sign}(\xi)$$

and

$$f_0''(\xi) = \frac{1}{t} > 0.$$

Thus, the only possile critical points $\xi \neq 0$ must satisfy

$$\xi = x + t\,\mathrm{sign}(\xi)$$

and must repreesnt local minima. There is, however, the point of non-differentiability at $\xi = 0$, so things are more complicated than these derivatives might suggest. To see what's going on, let us consider four cases.

Case a$_0$: $0 < t \leq x$. In this case, critical points $\xi < 0$ are not possible since $\xi = x + t\,\mathrm{sign}(\xi) < 0$ would mean $\xi = x - t \geq 0$.

On the other hand, f_0 does have a local minimum at $\xi = x + t\,\mathrm{sign}(\xi) = x + t > 0$, and it is easily checked that $f_0'(\xi) < 0$ for $\xi < 0$. We conclude that f_0 is strictly decreasing for $-\infty < \xi < x + t$ to a global minimum value

$$f_0(x+t) = \frac{t}{2} + 1 - x - t = 1 - x - \frac{t}{2}.$$

The function f_0 is strictly increasing for $x + t < \xi < \infty$. Typical behavior in this case is indicated in Figure 1.

Case b$_0$: $0 \leq x \leq t$.

Case c$_0$: $-t \leq x \leq 0$.

Case d$_0$: $x \leq -t < 0$.