Math 6342, Final Exam (practice) Name and section: ____________

The problems on this exam are related to the notes on the "main existence and uniqueness theorem" for linear elliptic PDE. These notes are posted at
`http://www.math.gatech.edu/ mccuan/courses/6342/existence.pdf`.

1. (33+1/3 points) (weak supersolutions)

A Define what is meant by a weak supersolution for the operator

$$Lu = -\sum_{i,j} D_i(a_{ij}D_j u) + \sum_j b_j D_j u + cu.$$

B Show that the expression $\inf\{M : (u^+ - M)^+ \in H_0^1(\mathcal{U})\}$ appearing in the weak maximum principle in the notes is the same as

$$\inf\{M : \max\{u, M\} - M \in H_0^1(\mathcal{U})\}.$$

Solution:

A A weak supersolution should satisfy $Lu \geq 0$ on $\mathcal{U}$ on some domain $\mathcal{U}$ in some weak sense. One such possible sense is the following:

$$B(u, v) \geq 0 \qquad \text{for all } v \in C_c^\infty(\mathcal{U}) \quad \text{with } v \geq 0,$$

where

$$B(u, v) = \sum_{i,j} \int_{\mathcal{U}} a_{ij} D_j u D_i v + \sum_j \int_{\mathcal{U}} b_j D_j u v + \int_{\mathcal{U}} cuv.$$

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B This problem is a bit trickier than it might first appear.

It might seem easy if you think that the expressions $(u^+ - M)^+$ and $\max\{u, M\} - M$ are equal. This is not true in general. It is true, however, when $M \geq 0$ which is what is important in the first set. To put this another way, we should first observe that

$$\inf\{M : (u^+ - M)^+ \in H_0^1(\mathcal{U})\} = \inf\{M \geq 0 : (u^+ - M)^+ \in H_0^1(\mathcal{U})\}.$$

It turns out that this is not easy to see. There are some notes on "Trace properties without boundary regularity" which give a proof. On the other hand, if we assume $\partial\mathcal{U}$ is C^1, then we can use the trace theorem (page 272 in Evans' book): If $M < 0$, then $u^+ - M \geq -M > 0$. We now use the fact (justified below) that $u^+ \in H_0^1(\mathcal{U})$. Once we know that, we can take a sequence of C_c^∞ functions which converge to u^+ in H_0^1. Denoting this sequence by ϕ_j, we see that $\psi_j = \phi_j - M$ is a sequence of C^∞ functions which converges to $u^+ - M$ and has value at least $-M > 0$ in a nieghborhood of $\partial\mathcal{U}$. In particular, the trace $T[u^+ - M] = \lim T\psi_j \geq -M > 0$ on $\partial\mathcal{U}$. On the other hand, if we assume $(u^+ - M)^+ = u^+ - M \in H_0^1(\mathcal{U})$ when $M < 0$, then we get a sequence in C_c^∞ converging in H^1 to $u^+ - M$. And again by the trace theorem, we have $T[u^+ - M] = 0$. This contradiction means $(u^+ - M)^+$ is never in $H_0^1(\mathcal{U})$ when $M < 0$ as desired.

It should be emphasized that we have used the assumption that $\partial\mathcal{U}$ is C^1 here. A more general discussion which doesn't use this boundary regularity is given in the notes.

For the application of the weak maximum principle to uniqueness, we would also like to know that $\sup_{\partial\mathcal{U}} u^+ \leq 0$ in the trace sense whenever $u \in H_0^1(\mathcal{U})$. This follows simply from the fact that $M = 0$ is in the set $\{M : (u^+ - M)^+ \in H_0^1(\mathcal{U})\}$. That is to say $u^+ \in H_0^1(\mathcal{U})$. This assertion, as mentioned above, is justified below.

Having gotten this (difficult) observation out of the way, the modification I intended you to find was: The set can be replaced with

$$\{M \geq 0 : \max\{u, M\} - M \in H_0^1(\mathcal{U})\}.$$

This, as mentioned above, is relatively easy to see.

In fact, if $u(x) \leq 0 \leq M$, then $u^+(x) = 0$ and $(u^+ - M)^+(x) = (-M)^+ = 0$ while $\max\{u(x), M\} = M$, so $\max\{u, M\} - M = M - M = 0$ too.

Similarly, $0 < u(x) \leq M$, then the value of $(u^+ - M)^+$ is 0 and the value of $\max\{u, M\} - M$ is also 0.

Finally, if $0 \leq M < u(x)$, then the value of $(u^+ - M)^+$ is $u(x) - M$ and the value of $\max\{u, M\} - M$ is also $u(x) - M$.

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So, we see that everything works as long as $M \geq 0$.

A technically correct (but much less instructive) modification is given by simply replacing u in the second set by u^+.

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2. (33+1/3 points) (weak supersolutions)

A Formulate a *weak minimum principle for weak supersolutions*. Be sure to include all necessary hypotheses and define all conditions which require special attention, e.g., what is the infemum of u on the boundary of a domain?

B Give a detailed proof of the uniqueness of weak solutions asserted in Corollary 1 of the notes.

Solution:

A Theorem 1 *(weak minimum principle for weak supersolutions) Assume $c \geq 0$. If $u \in H^1(\Omega)$ and $Lu \geq 0$ in the sense that*

$$B(u, v) \geq 0 \qquad \text{for all } v \in C_c^\infty(\Omega) \text{ with } v \geq 0,$$

then

$$\inf_\Omega u \geq \inf_{\partial\Omega} u^-$$

where the infemum on the left is the essential infemum defined by

$$\sup\{M : \text{measure}\{x : u(x) \leq M\} = 0\},$$

$u^- = \min\{u, 0\}$, *and the infemum on the right is taken in the trace sense:*

$$\sup\{M : (u^- + M)^- \in H_0^1(\Omega)\}.$$

B Let u and $\tilde{u} \in H_0^1$ be two weak solutions of $Lu = f$. Then $w = u - \tilde{u}$ is a weak solution of $Lw = 0$ in H_0^1. By the weak maximum principle,

$$\sup_\Omega u - \tilde{u} \leq \sup_{\partial\Omega} (u - \tilde{u})^+.$$

Looking at the definition

$$\sup_{\partial\Omega} (u - \tilde{u})^+ = \inf\{M : [(u - \tilde{u})^+ - M)^+ \in H_0^1(\mathcal{U})\}$$

and taking $M = 0$, we are asking the question: "Is $w^+ = (u - \tilde{u})^+ = \max\{u - \tilde{u}, 0\}$ in H_0^1?"

We know $w \in H_0^1$, and here is where we need to know $\sup_{\partial\Omega} w \leq 0$ in the trace sense.

Lemma If $w \in H_0^1(\mathcal{U})$, then $w^+ \in H_0^1(\mathcal{U})$.

Proof: It is shown in the notes on "Trace properties without boundary regularity" that $w^+ \in H^1$ with

$$Dw^+(x) = \begin{cases} Dw(x), & x \in \{\xi : w(\xi) > 0\} \\ 0, & x \in \{\xi : w(\xi) \leq 0\}. \end{cases}$$

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We can also take a sequence w_j of C_c^∞ functions converging to w in H^1. By the same application of the result in the notes $w_j^+ \in H^1$ and since w_j^+ has compact support in Ω, the standard regularization of $\eta_\epsilon * w_j^+$ is in $C_c^\infty(\mathcal{U})$ if ϵ is small enough. In particular, we see $w_j^+ \in H_0^1$. Moreover, given $\delta > 0$, we can take j large enough so that

$$\|w_j^+ - w^+\|_{H^1} < \delta/2$$

and $\epsilon > 0$ small enough, so that

$$\|\eta_\epsilon * w_j^+ - w_j^+\|_{H^1} < \delta/2.$$

It follows that there are C_c^∞ functions converging to w^+ in H^1. That is, $w^+ \in H_0^1(\mathcal{U})$. $\square$

We conclude that $w \leq 0$ by the weak maximum principle. Similarly, we have $w \geq 0$ by the weak minimum principle for supersolutions.

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3. (33+1/3 points) In the proof of the main existence theorem, we used the uniqueness which follows from the weak maximum/minimum principle to assert that the solution we obtained was unique. This required $c \geq 0$.

   For this problem, I want you to set aside the condition $c \geq 0$. As a consequence the first part of the proof of the main theorem is no longer valid. Nevertheless, we can still use the Fredholm theorem, and the argument showing existence of a solution when the second alternative holds is still perfectly valid.

   Assuming we are in the second case of the Fredholm alternative, give an alternative proof of uniqueness of the solution shown to exist in the notes. (Hint: Use the uniqueness asserted in the Fredhold alternative itself.)

Solution: First observe that $H_0^1(\mathcal{U}) \subset L^2(\mathcal{U})$. Therefore, if u and $\tilde{u}$ are both solutions according to the condition

$$B(u, v) = \langle f, v \rangle_{L^2} \qquad \text{for all } v \in H_0^1(\mathcal{U}),$$

then

$$\tilde{B}(u - \tilde{u}, v) = \langle \mu_0(u - \tilde{u}), v \rangle_{L^2} \qquad \text{for all } v \in H_0^1(\mathcal{U}).$$

That is, $\tilde{\Lambda}(\mu_0(u - \tilde{u})) = u - \tilde{u}$. Here it is crucial to think of $\tilde{f} = \mu_0(u - \tilde{u})$ as a given function in $L^2(\mathcal{U})$. From there we have

$$\tilde{\Lambda}\left(u - \tilde{u} - \frac{1}{\mu_0} I(u - \tilde{u})\right) = 0,$$

or

$$\left(\tilde{\Lambda} u - \frac{1}{\mu_0} \bar{I}\right)(u - \tilde{u}) = 0 = 0,$$

where $\bar{I}$ is the identity on L^2. Thus, by the second alternative, $u - \tilde{u}$ is the unique function ϕ in $L^2(\mathcal{U})$ such that $(\tilde{\Lambda} - \bar{I}/\mu_0)\phi = 0$. That is, $u - \tilde{u} = 0 \in L^2$.