Math 6342, Exam 2 (practice) Name and section: ______________

1. (20 points) (Weak/Strong Solutions) Let $\{a_{ij}\}$ be a collection of bounded coefficients, $f \in L^2(\Omega)$, and $u \in H_0^1(\Omega)$. Show that if

$$\int \sum a_{ij} D_j u D_i \eta = \int f\eta \qquad \forall\ \eta \in C_c^\infty(\Omega)$$

Then

$$\int \sum a_{ij} D_j u D_i v = \int f v \qquad \forall\ v \in H_0^1(\Omega).$$

Solution: Since H_0^1 is the closure of C_c^∞, there is a sequence $\eta_k \to v$ (in the H^1 norm), i.e.,

$$|\eta_k - v|_{H^1} \to 0 \qquad \text{as } k \to \infty.$$

Therefore, letting $M = \sup |a_{ij}|$ with the sup taken over all i, j, and $x \in \Omega$, we have

$$\begin{aligned}
\left| \int \sum a_{ij} D_j u D_i \eta_k - \int \sum a_{ij} D_j u D_i v \right| &\le M \sum \int |D_j u||D_i \eta_k - D_i v| \\
&\le M \sum_{i,j} |D_j u|_{L^2} |D_i \eta_k - D_i v|_{L^2} \\
&\le M \sum_j |D_j u|_{L^2} \sum_i |D_i \eta_k - D_i v|_{L^2} \\
&\le M |u|_{H^1} |\eta_k - v|_{H^1} \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left| \int f\eta_k - \int f v \right| &\le |f|_{L^2} |\eta_k - v|_{L^2} \\
&\le |f|_{H^1} |\eta_k - v|_{H^1} \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int \sum a_{i,j} D_j u D_i v = \lim_{k\to\infty} &\int \sum a_{i,j} D_j u D_i \eta_k \\
\lim_{k\to\infty} &\int f\eta_k \\
&\int f v.
\end{aligned}$$

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2. (20 points) Prove Reisz' lemma in Hilbert space: If W is a proper closed subspace of the Hilbert space $\mathcal{H}$, then there is a vector $\xi \in \mathcal{H}$ with $\|\xi\| = 1 = \text{dist}(\xi, W)$.

Give an example showing the condition that W is closed is needed in Reisz' result.

Solution: Since W is closed and proper, the orthogonal complement of W is a nontrivial subspace of $\mathcal{H}$. Let ξ be an element of the unit ball in $W^{\perp}$. For any $w \in W$ we have

$$\begin{aligned}\|\xi - w\| &= \sqrt{\|\xi\|^2 - 2\langle \xi, w\rangle + \|w\|^2} \\ &= \sqrt{\|\xi\|^2 + \|w\|^2} \\ &\geq \|\xi\| \\ &= 1.\end{aligned}$$

Therefore, $\text{dist}(\xi, W) \geq 1$. On the other hand, $0 \in W$ and $\|\xi - 0\| = 1$, so $\text{dist}(\xi, W) \leq 1$.

For the second part, take $\mathcal{H} = L^2[0, 1]$ and W to be the proper subspace of polynomials (or trigonometric polynomials). In this case, W is known to be dense in $\mathcal{H}$. That is, for any $\xi \in \mathcal{H}$, we have $\text{dist}(\xi, W) = 0$.

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3. (20 points) (solution operator)

(i) What is the form of a general *divergence form* second order linear partial differential operator?

(ii) What is the *Dirichlet problem* for a linear partial differential operator?

(iii) Given a linear partial differential operator L (as you have defined above) define what it means for $u \in H_0^1$ to be a solution of the *zero (homogeneous) boundary values* Dirichlet problem for L.

Solution:

(i) The form of a general divergence form second order linear partial differential operator is

$$Lu = -\sum_{i,j} D_i(a_{ij} D_j u) + \sum_j b_j D_j u + cu$$

where the natural/classical domain for L is $C^2(\mathcal{U})$ on some open set $\mathcal{U} \subset \mathbb{R}^n$, and the coefficients a_{ij}, b_j and c are all continuous functions on $\bar{\sqcap}$ with the $a_{ij} \in C^1(\mathcal{U})$ and symmetric.

(ii) The *Dirichlet problem* for L is

$$\begin{cases} Lu = f & \text{on } \mathcal{U} \\ u\big|_{\partial\mathcal{U}} = g \end{cases}$$

where $g \in C^0(\partial\mathcal{U})$ is given.

(iii) $u \in H_0^1(\mathcal{U})$ is a weak solution of

$$\begin{cases} Lu = f & \text{on } \mathcal{U} \\ u\big|_{\partial\mathcal{U}} = 0 \end{cases}$$

if $B(u, \phi) = \langle f, \phi \rangle_{L^2(\mathcal{U})}$ for all $\phi \in C_c^\infty(\mathcal{U})$ where $B : H_0^1(\mathcal{U}) \times H_0^1(\mathcal{U}) \to \mathbb{R}$ is a bilinear form given by

$$B(u, v) = \sum_{ij} \int_{\mathcal{U}} a_{ij} D_j u D_i v + \sum_j \int_{\mathcal{U}} b_j D_j u v + \int_{\mathcal{U}} cuv$$

with $D_j u$ and $D_i v$ representing first order weak derivatives of u and v respectively.

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The next two problems concern the linear partial differential operator

$$Lu = -\sum_{i,j} D_i(a_{ij} D_j u)$$

with $\{a_{ij}\} \in C_c^\infty(\mathcal{U}) \cap C^0(\bar{\mathcal{U}})$ a collection of smooth coefficients.

4. (20 points) (solution operator)

(i) Show there is a linear operator Λ which assigns to each $f \in L^2(\mathcal{U})$ the unique (weak) solution $u \in H_0^1(\mathcal{U})$ of the Dirichlet problem for L (with zero boundary values).

(ii) It is clear that the compact operator $\tilde{\Lambda} : L^2 \to L^2$ given by composing the natural compact embedding of $H_0^1(\mathcal{U})$ into $L^2(\mathcal{U})$ on the solution operator is one-to-one but *not* onto simply because $H_0^1 \neq L^2$. Show that a compact operator $\tilde{\Lambda} : \mathcal{H} \to \tilde{\mathcal{H}}$ of infinite dimensional Hilbert spaces is never one-to-one and onto. (Hint: Read the proof that 0 is in the resolvent spectrum on page 727 of Evans' book.)

Solution:

(i) According to the Lax-Milgram theorem, we only need to show that $B(u,v) = \sum \int a_{ij} D_j u D_i v$ is bounded and coercive. In fact,

$$\begin{aligned} |B(u,v)| &\leq \sum \int |a_{ij}||D_j u||D_i v| \\ &\leq A \sum_{ij} \|D_j u\|_{L^2} \|D_i v\|_{L^2} \\ &\leq A \left(\sum_j \|D_j u\|_{L^2} \right) \left(\sum_i \|D_i v\|_{L^2} \right) \end{aligned}$$

$$\leq A \|u\|_{H^1} \|v\|_{H^1}.$$

where

$$A = \sup_{i,j,x} |a_{ij}(x)| < \infty.$$

To see coercivity, we need the Poincaré inequality for $u \in H_0^1(\mathcal{U})$ which says there is some $C > 0$ for which

$$\|u\|_{L^2(\mathcal{U})} \leq C \|Du\|_{L^2(\mathcal{U})}.$$

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Then we see

$$
\begin{aligned}
B(u,u) &\geq \epsilon_0 \int |Du|^2 \\
&= \frac{\epsilon_0}{2} \int |Du|^2 + \frac{\epsilon_0}{2} \int |Du|^2 \\
&\geq \frac{\epsilon_0}{2} \int |Du|^2 + \frac{\epsilon_0}{2C} \int |u|^2 \\
&\geq \min\left\{\frac{\epsilon_0}{2}, \frac{\epsilon_0}{2C}\right\} \left(\int |Du|^2 + \int |u|^2 \right) \\
&\geq m \|u\|_{H^1}
\end{aligned}
$$

for some $m > 0$.

(ii) Assume $\tilde{\Lambda}$ is one-to-one and onto. Then the identity mapping $I : \mathcal{H} \to \mathcal{H}$ can be written as $I = \tilde{\Lambda}^{-1} \circ \tilde{\Lambda}$. We claim first that I is compact. To see this, let u_j be a bounded sequence in $\mathcal{H}$. Then there is a convergent subsequence $\tilde{\Lambda}(u_{j_k}) = v_k \to v$ in $\tilde{\mathcal{H}}$. By the open mapping theorem $\tilde{\Lambda}^{-1}$ is a bounded linear operator, so $I(u_{j_k}) = \tilde{\Lambda}^{-1}(v_k)$ also converges to $\tilde{\Lambda}^{-1}(v)$. Therefore, I is compact.

On the other hand, in an infinite dimensional Hilbert space, we can take $\{u_j\}$ to be an orthonormal sequence with $\|I(u_j) - I(u_k)\|^2 = \|u_j - u_k\|^2 = 2$. Therefore, it is impossible to find a convergent subsequence of $\{I(u_j)\}$, and we have a contradiction.

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5. (20 points) (solution operator)

(i) Show that the solution operator from the previous problem can be generalized: For each $\ell \in \mathcal{H}^*$ where $\mathcal{H} = H_0^1(\mathcal{U})$, there is a unique $u \in \mathcal{H}$ such that

$$B(u,v) = \ell(v) \qquad \text{for all } v \in \mathcal{H}$$

where B is the bilinear form associated with L.

(ii) Show that the solution operator $\Lambda : \mathcal{H}^* \to \mathcal{H}$ for the generalized problem is one-to-one and onto. Why does this not contradict the result of problem 4(ii)?

Solution:

(i) The application of the Lax-Milgram theorem works for any $\ell \in \mathcal{H}^*$; we just need L to be bounded and coercive, which we have in this case since only the top order terms are included.

(ii) If $u = \Lambda(\ell) = \Lambda(\tilde{\ell})$, then $\ell(v) = B(u,v) = \tilde{\ell}(v)$ for all $v \in \mathcal{H}$. This means that ℓ and $\tilde{\ell}$ are the same functional. Hence, Λ is one-to-one.

Let $u \in \mathcal{H}$. Then $\ell(v) = B(u,v) = \sum a_{ij} D_j u D_i v$ defines a bounded linear functional on $\mathcal{H}$. That is, $\ell \in \mathcal{H}^*$. Clearly, u is the solution of $B(u,v) = \ell(v)$ for all $v \in \mathcal{H}$, that is, $\Lambda(\ell) = u$. so Λ is onto.

There is no contradiction because we have not shown (and it is not true) that Λ is a compact operator.

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6. (10 points) (Bonus) Show that the solution operator $\Lambda : L^2 \to H^1_0$ as described in Problem 4 above is *not* onto.

Solution:

L^2 naturally embeds in $\mathcal{H}^*$ by $I[f](v) = \int fv = \langle f, v\rangle$ for $v \in \mathcal{H} = H^1_0(\mathcal{U})$. Thus, letting $\bar{\Lambda} : \mathcal{H}^* \to \mathcal{H}$ denote the generalized solution operator of problem 5, we can consider $\bar{\Lambda} \circ I$. Since I is one-to-one and $\bar{\Lambda}$ is one-to-one and onto, the question reduces to showing that I is not onto, i.e., we need to find a functional $\ell \in \mathcal{H}^*$ which does not come from integration against an L^2 function. This is pretty easy.

Let $\ell(v) = \int \phi D_j v$ where $\phi \in L^2$ is a function *without* a j-th weak derivative. Then assume

$$\ell(v) = \int fv \qquad \text{for all } v \in \mathcal{H}.$$

That is,

$$\int \phi D_j v = \int fv \qquad \text{for all } v \in \mathcal{H}.$$

In particular $-f$ is a weak j-th derivative for ϕ which contradicts what we know about ϕ.