Math 6342, Exam 2 (practice) Name and section: ____________________

1. (20 points) (Weak/Strong Solutions) Let $\{a_{ij}\}$ be a collection of bounded coefficients, $f \in L^2(\Omega)$, and $u \in H_0^1(\Omega)$. Show that if

$$\int \sum a_{ij} D_j u D_i \eta = \int f \eta \qquad \forall\ \eta \in C_c^\infty(\Omega)$$

Then

$$\int \sum a_{ij} D_j u D_i v = \int f v \qquad \forall\ v \in H_0^1(\Omega).$$

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2. (20 points) Prove Reisz' lemma in Hilbert space: If W is a proper closed subspace of the Hilbert space $\mathcal{H}$, then there is a vector $\xi \in \mathcal{H}$ with $\|\xi\| = 1 = \operatorname{dist}(\xi, W)$.

Give an example showing the condition that W is closed is needed in Reisz' result.

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3. (20 points) (solution operator)

(i) What is the form of a general *divergence form* second order linear partial differential operator?

(ii) What is the *Dirichlet problem* for a linear partial differential operator?

(iii) Given a linear partial differential operator L (as you have defined above) define what it means for $u \in H_0^1$ to be a solution of the *zero (homogeneous) boundary values* Dirichlet problem for L.

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The next two problems concern the linear partial differential operator

$$Lu = -\sum_{i,j} D_i(a_{ij} D_j u)$$

with $\{a_{ij}\} \in C_c^\infty(\mathcal{U}) \cap C^0(\bar{\mathcal{U}})$ a collection of smooth coefficients.

4. (20 points) (solution operator)

(i) Show there is a linear operator Λ which assigns to each $f \in L^2(\mathcal{U})$ the unique (weak) solution $u \in H_0^1(\mathcal{U})$ of the Dirichlet problem for L (with zero boundary values).

(ii) It is clear that the compact operator $\tilde{\Lambda} : L^2 \to L^2$ given by composing the natural compact embedding of $H_0^1(\mathcal{U})$ into $L^2(\mathcal{U})$ on the solution operator is one-to-one but *not* onto simply because $H_0^1 \neq L^2$. Show that a compact operator $\tilde{\Lambda} : \mathcal{H} \to \tilde{\mathcal{H}}$ of infinite dimensional Hilbert spaces is never one-to-one and onto. (Hint: Read the proof that 0 is in the resolvent spectrum on page 727 of Evans' book.)

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5. (20 points) (solution operator)

(i) Show that the solution operator from the previous problem can be generalized: For each $\ell \in \mathcal{H}^*$ where $\mathcal{H} = H_0^1(\mathcal{U})$, there is a unique $u \in \mathcal{H}$ such that

$$B(u, v) = \ell(v) \qquad \text{for all } v \in \mathcal{H}$$

where B is the bilinear form associated with L.

(ii) Show that the solution operator $\Lambda : \mathcal{H}^* \to \mathcal{H}$ for the generalized problem is one-to-one and onto. Why does this not contradict the result of problem 4(ii)?

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6. (10 points) (Bonus) Show that the solution operator $\Lambda : L^2 \to H_0^1$ as described in Problem 4 above is *not* onto.