Math 6342, Exam 1 (practice) Name and section: ____________________

1. (25 points) (Hamilton-Jacobi Equation) Consider the initial value problem

$$\begin{cases} u_t + |u_x|^3 = 0 \text{ on } \mathbb{R} \times (0,\infty) \\ u(x,0) = |x|. \end{cases}$$

(i) Write down the Hopf-Lax formula for a solution of this IVP.

(ii) Evaluate the Hopf-Lax formula and verify that it gives a solution.

Solution:

(i) The Hopf-Lax formula for this IVP is given by

$$u(x,t) = \min_{\xi \in \mathbb{R}^n} \left\{ tL\left(\frac{x-\xi}{t}\right) + |\xi| \right\}$$

where $L = H^*$ is the convex dual of the Hamiltonian $H(p) = |p|^3$ appearing in the PDE. The formula for the convex dual in this case is given by

$$L(v) = \max_p \{pv - |p|^3\}.$$

The derivative with respect to p of the expression we need to maximize is $v - 3p|p|$ which vanishes for $p = v/\sqrt{3|v|}$ unless $v = 0$, in which case the minimum is clearly at $p = 0$. The minimum value in the nontrivial case is

$$L(v) = \frac{v^2}{\sqrt{3|v|}} - \frac{v^2}{3\sqrt{3|v|}} = \frac{2}{3\sqrt{3}}|v|\sqrt{|v|}.$$

Thus, the Hopf-Lax formula is

$$\begin{aligned} u(x,t) &= \min_{\xi \in \mathbb{R}^n} \left\{ \frac{2t}{3\sqrt{3}} \left(\frac{|x-\xi|}{t}\right)^{3/2} + |\xi| \right\} \\ &= \min_{\xi \in \mathbb{R}^n} \frac{2}{3\sqrt{3t}}|x-\xi|^{3/2} + |\xi|. \end{aligned}$$

(ii) To find this minimum, we take the derivative with respect to ξ and attempt to solve:

$$-\frac{x-\xi}{\sqrt{3t|x-\xi|}} + \frac{\xi}{|\xi|} = 0.$$

This will be OK as long as $|\xi| \neq 0$ and, in that case, $|x-\xi| = 3t$, and the minimum simplifies to $u(x,t) = 2t + |\xi|$. On the other hand, solving directly we find $\xi = x \pm 3t$. Thus, checking cases, we find the associated minima would be $2t + |x \pm 3t|$ if $\pm x \leq 0$.

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If $|\xi| = 0$, then u becomes $2|x|^{3/2}/(3\sqrt{3t})$. Thus, we consider cases: If $x \le -3t$, then we need to compare $2\sqrt{-x^3}/(3\sqrt{3t})$ with $-x-t$. Noting that

$$\frac{\partial}{\partial t}\left(-x - t - \frac{2\sqrt{-x^3}}{3\sqrt{3t}}\right) = -1 + \frac{1}{3\sqrt{3}}\left(\frac{-x}{t}\right)^{3/2}$$

vanishes only for $x = -3t$ and $-x-t$ is clearly smaller when t is close to zero, we see that this must be the min value in the entire region.

The next case is $-3t \le x \le 0$. Notice that both alternatives have the same value $2t$ along the line $x = -3t$. Thus, it is enough to compute

$$\frac{\partial}{\partial t}\left(x + 5t - \frac{2\sqrt{-x^3}}{3\sqrt{3t}}\right) = 5 + \frac{1}{3\sqrt{3}}\left(\frac{-x}{t}\right)^{3/2} > 0.$$

Similar analysis leads to the conclusion

$$u(x,t) = \begin{cases} -x-t, & x \le -3t \\ 2\sqrt{-x^3}/(3\sqrt{3t}), & -3t \le x \le 0 \\ 2\sqrt{x^3}/(3\sqrt{3t}), & 0 \le x \le 3t \\ x-t, & 3t \le x. \end{cases}$$

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2. (25 points) (separation of variables) Solve the initial/boundary value problem for the heat equation

$$\begin{cases} u_t = \Delta u \quad \text{on } (-1,1)\times(0,\infty) \\ u(\pm 1,t) = 0 \quad \text{for } t \geq 0, \\ u(x,0) = 1-|x| \quad \text{for } |x| \leq 1. \end{cases}$$

Solution: Setting $u = A(x)B(t)$, we find separated equations $A'' = -\lambda A$ and $B' = -\lambda B$ with boundary conditions $A(\pm 1) = 0$. If $\lambda > 0$, then $A = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x$, and the boundary conditions give $a\cos\sqrt{\lambda} \pm b\sin\sqrt{\lambda} = 0$, or $a\cos\sqrt{\lambda} = b\sin\sqrt{\lambda} = 0$. If $a \neq 0$, then we get

$$\lambda = \lambda_j = \left(\frac{\pi}{2} + j\pi\right)^2 \quad \text{for } j = 0, 1, \ldots$$

and $b = 0$. If $a = 0$, then we can assume $\sin\sqrt{\lambda} = 0$, so

$$\lambda = \tilde{\lambda}_j = j^2\pi^2 \quad \text{for } j = 1, 2, \ldots$$

In view of the B equation, we get separated variables solutions

$$u_j(x,t) = e^{-\lambda_j t}\cos\left(\frac{2j+1}{2}\pi x\right) \quad \text{and} \quad \tilde{u}_j(x,t) = e^{-\tilde{\lambda}_j t}\sin(j\pi x).$$

If $\lambda = 0$, then $A = ax + b$, but then $\pm a + b = 0$, so $a = b = 0$, and we get no eigenfunctions. Similarly, if $\lambda = -\mu < 0$, then $A = a\cosh\sqrt{\mu}x + b\sinh\sqrt{\mu}x$, and $a\cos\sqrt{\mu} \pm b\sinh\sqrt{\mu} = 0$ so that $a\cos\sqrt{\mu} = 0 = b\sinh\sqrt{\mu}$. Since $\cos\sqrt{\mu} \neq 0$, we see that $a = 0$. Then either $b = 0$ or $\mu = 0$. In either case, we get no eigenfunctions for $\lambda < 0$.

We next try a superposition of separated variables solutions:

$$u(x,t) = \sum_{j=0}^{\infty} a_j e^{-\lambda_j t}\cos\left(\frac{2j+1}{2}\pi x\right) + \sum_{j=1}^{\infty} \tilde{a}_j e^{-\tilde{\lambda}_j t}\sin(j\pi x).$$

From the initial condition we will need

$$a_j \int_{-1}^{1} \cos^2\left(\frac{2j+1}{2}\pi x\right)\,dx = \int_{-1}^{1} (1-|x|)\cos\left(\frac{2j+1}{2}\pi x\right)\,dx$$

and

$$\tilde{a}_j \int_{-1}^{1} \sin^2\left(\frac{2j+1}{2}\pi x\right)\,dx = \int_{-1}^{1} (1-|x|)\sin(j\pi x)\,dx = 0.$$

That is,

$$a_j = \frac{8}{(2j+1)^2\pi^2}.$$

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Thus, we can try

$$u(x,t) = \frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}e^{-\left(\frac{2j+1}{2}\right)^2\pi^2 t}\cos\left(\frac{2j+1}{2}\pi x\right).$$

Convergence for all x and t is clear, and it is easily checked (using term by term differentiation) that $u \in C^\infty(\mathbb{R}\times(0,\infty))$ and satisfies the PDE. The boundary conditions also clearly hold. It is also true that

$$\frac{8}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\cos\left(\frac{2j+1}{2}\pi x\right) = 1-|x|.$$

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3. (25 points) (Fourier Transform) Consider the Cauchy problem for the wave equation on $\mathbb{R}$:

$$\begin{cases} u_{tt} = \Delta u \quad \text{on } \mathbb{R} \times (0, \infty) \\ u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \\ u_t(x,0) = 0 \quad \text{for } x \in \mathbb{R}, \end{cases}$$

where

$$u_0(x) = \begin{cases} 1 - |x| \quad \text{for } |x| \leq 1 \\ 0 \quad \text{for } |x| \geq 1. \end{cases}$$

Let

$$\hat{u}(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{x \in \mathbb{R}} e^{-i\xi x} u(x)$$

be the spatial Fourier transform of u.

(i) Find an initial value problem satisfied by $\hat{u}$.

(ii) Determine $\hat{u}(\xi, t)$.

Solution:

(i)

$$\hat{u}_{tt} = \frac{1}{\sqrt{2\pi}} \int_{x \in \mathbb{R}} e^{-i\xi x} u_{tt} = \frac{1}{\sqrt{2\pi}} \int_{x \in \mathbb{R}} e^{-i\xi x} u_{xx} = -\xi^2 \hat{u}.$$

From the initial condition for u it's clear that $\hat{u}_t(\xi, 0) = 0$, but $\hat{u}(\xi, 0)$ must be computed. In fact,

$$\hat{u}(\xi, 0) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\xi x} (1 - |x|)\, dx = \frac{1}{\xi^2 \sqrt{2\pi}} e^{-i\xi} \left(e^{i\xi} - 1\right) = \sqrt{\frac{2}{\pi}} \left(\frac{\cos\xi - 1}{\xi^2}\right).$$

Thus, the initial value problem for $\hat{u}$ (with parameter ξ) is:

$$\begin{cases} \hat{u}_{tt} = -\xi^2 \hat{u} \quad \text{for } 0 < t \\ \hat{u}(\xi, 0) = \sqrt{\frac{2}{\pi}} \left(\frac{\cos\xi - 1}{\xi^2}\right) \\ \hat{u}_t(\xi, 0) = 0. \end{cases}$$

(ii) The solution of the ODE has the form $\hat{u} = a \cos \xi t + b \sin \xi t$. Differentiating with respect to t and using the second initial condition, we find $b = 0$. The other initial condition implies

$$\hat{u}(\xi, t) = \sqrt{\frac{2}{\pi}} \left(\frac{\cos\xi - 1}{\xi^2}\right) \cos \xi t.$$

The next step would be to use the inverse Fourier transform to obtain a special case of d'Alembert's solution.

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4. (25 points) (5.10.2) Prove the interpolation inequality

$$|u|_{C^\gamma} \le |u|_{C^\beta}^{\frac{1-\gamma}{1-\beta}} |u|_{C^{0,1}}^{\frac{\gamma-\beta}{1-\beta}}$$

for any Lipschitz function u and any β and γ with $0 < \beta < \gamma \le 1$.

Solution: We are asked to show

$$|u|_\infty + \sup \frac{|u(x)-u(\xi)|}{|x-\xi|^\gamma} \le \left(|u|_\infty + \sup \frac{|u(x)-u(\xi)|}{|x-\xi|^\beta}\right)^{1-\lambda} \left(|u|_\infty + \sup \frac{|u(x)-u(\xi)|}{|x-\xi|}\right)^\lambda$$

where $\lambda = (\gamma-\beta)/(1-\beta)$ is between 0 and 1. Equivalently, setting $A = |u|_\infty$ and taking a logarithm, we need

$$\log\left(A + \sup \frac{|u(x)-u(\xi)|}{|x-\xi|^\gamma}\right) \le (1-\lambda)\log\left(A + \sup \frac{|u(x)-u(\xi)|}{|x-\xi|^\beta}\right) + \lambda \log\left(A + \sup \frac{|u(x)-u(\xi)|}{|x-\xi|}\right).$$

Let $x \ne \xi$ be both fixed in $\mathcal{U}$, set $b = |u(x)-u(\xi)|$, $c = |x-\xi|$, and consider the function $f(p) = \log(A + b/c^p)$ for $0 < p \le 1$. Notice that

$$f'(p) = \frac{-pb/c^{p+1}}{A + b/c^p} = -\frac{bp}{Ac^{p+1}+bc} \quad \text{and} \quad f''(p) = -\frac{b(Ac^{p+1}+bc) - bp(p+1)Ac^p}{(Ac^{p+1}+bc)^2} = \frac{b^2c}{(Ac^{p+1}+bc)^2} \ge$$

Thus, f is convex. In particular, $f((1-\lambda)\beta+\lambda) \le (1-\lambda)f(\beta) + \lambda f(1)$. Since $(1-\lambda)\beta + \lambda = \gamma$, this means

$$\log\left(A + \frac{|u(x)-u(\xi)|}{|x-\xi|^\gamma}\right) \le (1-\lambda)\log\left(A + \frac{|u(x)-u(\xi)|}{|x-\xi|^\beta}\right) + \lambda \log\left(A + \frac{|u(x)-u(\xi)|}{|x-\xi|}\right)$$

holds for every particular $x \ne \xi$. Taking the supremum of $x \ne \xi$ on both sides and noting that the supremum of a product is less than or equal to the product of the suprema of the factors, we obtain the desired inequality.