Math 6342, Exam 1 (practice) Name and section: __________________

1. (25 points) (Hamilton-Jacobi Equation) Consider the initial value problem

$$\begin{cases} u_t + |u_x|^3 = 0 \text{ on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = |x|. \end{cases}$$

(i) Write down the Hopf-Lax formula for a solution of this IVP.

(ii) Evaluate the Hopf-Lax formula and verify that it gives a solution.

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2. (25 points) (separation of variables) Solve the initial/boundary value problem for the heat equation

$$\begin{cases} u_t = \Delta u \quad \text{on } (-1,1)\times(0,\infty) \\ u(\pm 1, t) = 0 \quad \text{for } t \geq 0, \\ u(x,0) = 1 - |x| \quad \text{for } |x| \leq 1. \end{cases}$$

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3. (25 points) (Fourier Transform) Consider the Cauchy problem for the wave equation on $\mathbb{R}$:

$$\begin{cases} u_{tt} = \Delta u \quad \text{on } \mathbb{R} \times (0, \infty) \\ u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \\ u_t(x,0) = 0 \quad \text{for } x \in \mathbb{R}, \end{cases}$$

where

$$u_0(x) = \begin{cases} 1 - |x| \quad \text{for } |x| \leq 1 \\ 0 \quad \text{for } |x| \geq 1. \end{cases}$$

Let

$$\hat{u}(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{x \in \mathbb{R}} e^{-i\xi x} u(x)$$

be the spatial Fourier transform of u.

(i) Find an initial value problem satisfied by $\hat{u}$.

(ii) Determine $\hat{u}(\xi, t)$.

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4. (25 points) (5.10.2) Prove the interpolation inequality

$$|u|_{C^{\gamma}} \leq |u|_{C^{\beta}}^{\frac{1-\gamma}{1-\beta}} |u|_{C^{0,1}}^{\frac{\gamma-\beta}{1-\beta}}$$

for any Lipschitz function u and any β and γ with $0 < \beta < \gamma \leq 1$.