Math 6342 A Hamilton-Jacobi Problem

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Let

$$u_0(x) = \begin{cases} 1 - |x|, & \text{if } |x| \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Consider the following variational problem: Minimize

$$\mathcal{F}[\mathbf{w}] = \int_a^b \left[\frac{\|\mathbf{w}'(t)\|^2}{2} - \alpha w_3(t) \right] dt + u_0(\mathbf{w}(a))$$

over the admissible class

$$\mathcal{A} = \left\{ \mathbf{w} = (w_1, w_2, w_3) \in C^2[a,b] : \mathbf{w}(b) = \mathbf{x}^* \right\}$$

where α is a positive constant.

(a) Find a formula $\mathbf{w}_0 = \mathbf{w}_0(t; \mathbf{x}^*, b)$ for the minimizer. Hint: Consider the minimization problem for $\mathcal{G}[\mathbf{w}] = \mathcal{F}[w] - u_0(\mathbf{w}(a))$ first.

(b) Assume $a = \alpha = 0$, and let $u(\mathbf{x}^*, t) = \mathcal{F}[\mathbf{w}_0]$ be the minimum value obtained above. Show that

$$u(x,t) = \min_{\xi \in \mathbb{R}^3} \left\{ \frac{|x - \xi|^2}{2t} + u_0(\xi) \right\}.$$

(c) Find the Legendre transform L^* of $L(v) = \|v\|^2/2$.

(d) Set $H(p) = L^*(p)$ with L given above. Show that $u(x,t)$ is a weak solution of the IVP

$$\begin{cases} u_t + H(Du) = 0, & \text{on } \mathbb{R}^3 \times (0, \infty) \\ u(x,0) = u_0(x). \end{cases}$$

This is a pretty tricky problem. Let's take it slowly and start with a task which is implicit in part (b). There is a calculus problem there which is pretty interesting on its own. That problem is:

Given $(x, t) \in \mathbb{R}^3 \times (0, \infty)$ *fixed, minimize*

$$f(\xi) = \frac{|x - \xi|^2}{2t} + u_0(\xi)$$

over $\xi \in \mathbb{R}^3$.

To get the hang of how to do such problems, it is often a good idea to consider easier ones first, and an obvious notion of "easier" in this case is "lower dimensional." In accord with this strategy, let's minimize the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(\xi) = \frac{(\xi - x)^2}{2t} + u_0(\xi)$$

where we take u_0 with the same definiton but domain $\mathbb{R}$. Notice that

$$f(\xi) = \begin{cases} f_0(\xi) & \text{for } |\xi| \leq 1 \\ f_1(\xi) & \text{for } |\xi| \geq 1 \end{cases}$$

where

$$f_0(\xi) = \frac{(\xi - x)^2}{2t} + 1 - |\xi|$$

and

$$f_1(\xi) = \frac{(\xi - x)^2}{2t}.$$

Since $1 - |\xi| = 0$ when $|\xi| = 1$, it is clear that f is continuous. In fact, if we put $|\xi - x|^2$ back in place of $(\xi - x)^2$, these same observations would hold in the higher dimensional case.

Now, the function f_1 has a kind of strong uniform coercivity. To be precise, if $|\xi| \geq r = \max\{1, |x|\}$, we can be sure that $f(\xi)$ is increasing in $|\xi|$ and f_1, and hence f, attains a minimum somewhere on the closure of $B_r(0)$.

As a special case of this coercivity property, we note that when $|x| \leq 1$, the minimum of f is always taken on the closure of $B_1(0)$ which is just $[-1, 1]$ in this 1-D case.

We also observe that f_1 has a unique global min at $\xi = x$ with $f(x) = 0$ and is strictly convex everywhere. Since $f \geq 0$, we see that this also determines a global min for f at $\xi = x$ whenever $|x| \geq 1$.

The other function f_0 is more complicated—and, in some sense, that is where the real action is happening. Of course, f_0 is just f_1, which we understand (the graph is a parabola whose vertex is at $\xi = x$ and has aspect ratio determined by t with small t meaning "taller" and larger t meaning "flatter") with $u_0(\xi)$ added. The sum is differentiable everywhere except $\xi = 0$, and f_0 is differentiable from the left and right even there. In fact,

$$\frac{df_0}{d\xi}(0^-) = -\frac{x}{t} + 1,$$

$$\frac{df_0}{d\xi}(0^+) = -\frac{x}{t} - 1,$$

and in general for $\xi \neq 0$

$$f_0'(\xi) = \frac{\xi - x}{t} - \operatorname{sign}(\xi)$$

and

$$f_0''(\xi) = \frac{1}{t} > 0.$$

Since $t > 0$, the one sided derivatives tell us that there can never be even a local min at $\xi = 0$. (If $x \geq 0$, then the right derivative is negative, and if $x \leq 0$, then the left derivative is positive.)

Thus, the only possible local minimum points must be critical points $\xi \neq 0$ and must satisfy

$$\xi = x + t \operatorname{sign}(\xi)$$

In fact, any such points must represent local minima.

Note that the two "pieces" of the graph of f_0 to the left and right of $\xi = 0$ are both parabolas (both of the same aspect ratio as the graph of f_1) meeting continuously at $\xi = 0$. If the left derivative at $\xi = 0$ is nonpositive, i.e., $x \geq t$, then f_0 has a unique global minimum at the unique critical point on the right at $\xi = x + t$.

Similarly, if the derivative at $\xi = 0$ on the right is positive ($x \leq -t$), then f_0 has a unique global minimum at $\xi = x - t$.

The most complicated case is when the left derivative is positive and the right derivative is negative, i.e., there is a local max at $\xi = 0$, i.e., $-t < x < t$. In that case, there are two local minima at $\xi = x \pm t$, and we just have to compare to see which one is lower. In fact, $f_0(x \pm t) = t/2 + 1 - |x \pm t|$. Since $x + t > 0$ and $x - t < 0$, we see that $f_0(x + t) = 1 - x - t/2$ and

$f(x-t) = 1 + x - t/2$. Clearly, if the center x of the original parabola has $x > 0$, then the min is on the right, and if the center is $x > 0$, then the min is on the left. If $x = 0$, then we have the same value on each side.

So, I think we understand f_0 as well as f_1. It remains to analyze the minimum when we put all this together in the conditional definition of f. One basic question which arises is whether or not the local minima of f_0 fall inside the interval $|\xi| \le 1$ or outside. This consideration leads to the following cases.

1. $0 < t \le x$.

   (a) $t \le 1 - x$ (and $0 < x < 1$)
   Here $x + t \le 1$, so $x < 1$ and the global min occurs in $[-1, 1]$. It must be at $\xi = x + t$ with value
   $$f(x+t) = 1 - x - \frac{t}{2}.$$

   (b) $1 - t \le x \le 1$ (and $1/2 < x < 1$)
   Here $x + t > 1$, but the global min must still occur on $[-1, 1]$. Since f_0 is nonincreasing across $[-1, 1]$, the min must occur at $\xi = 1$ with value
   $$f(1) = \frac{(1-x)^2}{2t}.$$

   (c) $x \ge 1$
   As mentioned above the min occurs at $\xi = x$ with value
   $$f(x) = 0.$$

2. $0 \le x \le t$

   (a) $x \le t \le 1 - x$ (and $0 \le x \le 1/2$)
   In this case, $x \le 1$ so the minimum occurs in $[-1, 1]$ and it occurs at the right local min of f_0 with value
   $$f(x+t) = 1 - x - \frac{t}{2}.$$

(b) $t \geq \max\{x, 1-x\}$ and $0 \leq x \leq 1$.

This is probably the most complicated case. Here the absolute min of f_0 occurs outside $[-1,1]$ and $x \leq 1$, so there is a local min at $\xi = 1$, but there is another local min on the left in $[-1,1]$ and $\xi = x - t$. We claim that the global min occurs at $\xi = 1$. To see this, first note that the two values are

$$f(x-t) = 1 + x - \frac{t}{2} \quad \text{and} \quad f(1) = \frac{(1-x)^2}{2t}.$$

Remembering that the left and right "parts" of the graph of f_0 are parts of parabolas of the same shape, we observe that

$$f(x_t) = f_0(0) - \frac{(x-t)^2}{2t}$$

while

$$f(1) = f_0(0) - \left[\frac{(x+t)^2}{2t} - \frac{(x+t-1)^2}{2t}\right].$$

Thus, the desired inequality reduces to showing

$$\left[\frac{(x+t)^2}{2t} - \frac{(x+t-1)^2}{2t}\right] \geq \frac{(x-t)^2}{2t}.$$

In fact, since $\tau \mapsto \tau^2$ is increasing and convex for $\tau \geq 0$ while $x + t - 1 \geq 0$ and $0 < t - x \leq 1$, we find

$$(x+t)^2 - (x+t-1)^2 \geq 1^2 - 0^2 \geq (t-x)^2 - 0^2.$$

Therefore, the minimum value is

$$f(1) = \frac{(1-x)^2}{2t}.$$

(c) $x \geq 1$

In this case, the min value is

$$f(x) = 0.$$

3. $-t \leq x \leq 0$.

This is similar the case 2 above, except everything "shifts left."

(a) $t \leq 1 + x$ (and $-1 < x < 0$)
The global min occurs at $\xi = x - t$ with value

$$f(x-t) = 1 + x - \frac{t}{2}.$$

(b) $1 - t \leq x \leq 1$ (and $-1 < x < -1/2$)
The global min occurs at $\xi = -1$ with value

$$f(-1) = \frac{(1+x)^2}{2t}.$$

(c) $x \leq -1$
The global min occurs at $\xi = x$ with value

$$f(x) = 0.$$

4. $0 \leq -x \leq t$

(a) $x \leq t \leq 1 + x$ (and $-1/2 \leq x \leq 0$)
The global minimum is

$$f(x-t) = 1 + x - \frac{t}{2}.$$

(b) $t \geq \max\{-x, 1 + x\}$ and $-1 \leq x \leq 0$.
The minimum value is

$$f(-1) = \frac{(1+x)^2}{2t}.$$

(c) $x \leq -1$
The min value is

$$f(x) = 0.$$

At this point, we have minimized f (which took twelve cases!), and we can reorganize and consider the minimum value as a function of $(x, t) \in \mathbb{R} \times (0, \infty)$:

$$u(x,t) = \begin{cases} 1 - |x| - t/2, & t \leq u_0(x) \\ (1-|x|)^2/(2t), & t \geq u_0(x),\ |x| \leq 1 \\ 0, & |x| \geq 1. \end{cases}$$

This function has some interesting properties.

In order to describe some of these properties, let us consider part (c) of the original problem:

Find

$$L^*(p) = \sup_{v \in \mathbb{R}^n} \left(p \cdot v - \frac{|v|^2}{2} \right),$$

This is relatively easy in any dimension. It is, again, a finite dimensional calculus problem. Fortunately, this problem doesn't involve u_0! Critical points are easy to find:

$$D\left(p \cdot v - \frac{|v|^2}{2} \right) = p - v \quad \text{and} \quad D^2\left(p \cdot v - \frac{|v|^2}{2} \right) = -I.$$

Thus, there is a unique max at $v = p$, and

$$L^*(p) = p \cdot p - \frac{|p|^2}{2} = \frac{|p|^2}{2}.$$

This completes part (c) and tells us that the IVP of part (d) is

$$\begin{cases} u_t + |Du|^2/2 = 0 & \text{on } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = u_0(x). \end{cases} \tag{1}$$

When $n = 1$, the function we obtained above in part (b) is a piecewise smooth piecewise solution of this problem. That's the first interesting thing about u.

The second thing is the regularity of u. We see that $u \in C^0(\mathbb{R} \times (0, \infty))$, but there is a distinct singularity along $x = 0$. On the other hand, the restriction of u to any domain which excludes $x = 0$ is C^1. If we consider the profile of $x \mapsto u(x, t)$, we see that as it evolves, the corner at $x = 0$ is preserved, but the corners at $x = \pm 1$ are smoothed, i.e., they vanish. This demonstrates the *semi-concavity* due to the uniform convexity of H (Lemma 4):

> *The concave corner propagates forward, but the convex corners become C^1 smooth.*

This is the characteristic regularizing associated with the Hamilton-Jacobi operator.

Next, let us return to the minimization problem of part (b) and consider generalizing what we have done. Before increasing the dimension, let's see what happens if we append a linear term:

$$u(x, t) = \min_{\xi \in \mathbb{R}} \left\{ \frac{|x - \xi|^2}{2t} + u_0(\xi) + m\xi \right\}.$$

Thinking of (x,t) as fixed, we can write

$$f(\xi) = \frac{(x-mt-\xi)^2}{2t} + u_0(\xi) + m\left(x - \frac{t}{2}\right).$$

Since x and t are fixed, it is evident that our previous (twelve) cases apply to minimize this function, and we find

1. $0 < t \le x - mt$, i.e., $(1+m)t \le x$.

   (a) $t \le 1 - x + mt$ or $(1-m)t \le 1 - x$.
   The global min occurs at $\xi = x - mt + t$ with value

   $$f(x + (1-m)t) = 1 - x + mt - \frac{t}{2}m\left(x - \frac{t}{2}\right).$$

   (b) $1 - t \le x - mt \le 1$.
   The global min occurs at $\xi = 1$ with value

   $$f(1) = \frac{(1-x+mt)^2}{2t} + m\left(x - \frac{t}{2}\right).$$

   (c) $x - mt \ge 1$
   The global min occurs at $\xi = x - mt$ with value

   $$f(x - mt) = 0 + m\left(x - \frac{t}{2}\right).$$

   It is interesting that the min value does not remain zero.

2. $0 \le x - mt \le t$

   (a) $x - mt \le t \le 1 - x + mt$
   The global min occurs at $\xi = x - mt + t$ with value

   $$f(x - mt + t) = 1 - x + mt - \frac{t}{2} + m\left(x - \frac{t}{2}\right).$$

   (b) $t \ge \max\{x - mt, 1 - x + mt\}$ and $0 \le x - mt \le 1$.
   The minimum value is

   $$f(1) = \frac{(1-x+mt)^2}{2t} + m\left(x - \frac{t}{2}\right).$$

(c) $x - mt \geq 1$
In this case, the min value is

$$f(x) = 0 + m\left(x - \frac{t}{2}\right).$$

3. $-t \leq x - mt \leq 0$.

(a) $t \leq 1 + x - mt$
The global min occurs at $\xi = x - mt - t$ with value

$$f(x - mt - t) = 1 + x - mt - \frac{t}{2} + m\left(x - \frac{t}{2}\right).$$

(b) $1 - t \leq x - mt \leq 1$
The global min occurs at $\xi = -1$ with value

$$f(-1) = \frac{(1 + x - mt)^2}{2t} + m\left(x - \frac{t}{2}\right).$$

(c) $x - mt \leq -1$
The global min occurs at $\xi = x - mt$ with value

$$f(x - mt) = 0 + m\left(x - \frac{t}{2}\right).$$

4. $0 \leq -x + mt \leq t$

(a) $x - mt \leq t \leq 1 + x - mt$
The global minimum is

$$f(x - mt - t) = 1 + x - mt - \frac{t}{2} + m\left(x - \frac{t}{2}\right).$$

(b) $t \geq \max\{-x + mt, 1 + x - mt\}$ and $-1 \leq x - mt \leq 0$.
The minimum value is

$$f(-1) = \frac{(1 + x - mt)^2}{2t} + m\left(x - \frac{t}{2}\right).$$

(c) $x - mt \leq -1$

The min value is

$$f(x) = 0 + m\left(x - \frac{t}{2}\right).$$

We now consider dimension $n = 2$ for the minimization problem (b).

Lemma 1 *1. If $|x| \leq 1$, then f attains all minima on $\overline{B_1(0)}$.*

2. If $|x| \geq 1$, then f has a global min at $\xi = x$ with $f(x) = 0$.

Proof: Since $f \geq 0$, the second assertion is obvious.

For the first assertion, observe that for each $v \in \mathbb{S}^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$, there is a unique $t \geq 0$ with $|x + tv| = 1$. Note, furthermore, that the value of $f(x + tv) = t/2$. This is obviously minimized when t is least. That is, when v is the unit vector pointing in the direction of the point on $\mathbb{S}^1$ closest to x. We have shown:

When $x \in \overline{B_1(0)}$

$$\min_{\xi \in \mathbb{S}^2} f(\xi) = \frac{(1 - |x|)^2}{2t}.$$

Next, we take a directional derivative in the direction v at the point $\xi = x + tv \in \mathbb{S}^1$:

$$D_v f(x + tv) = D_v f_1(x + tv)$$

where $f_1(\xi) = |x - \xi|^2/(2t)$ as in the 1-D case. We find

$$D_v f(x + tv) = D_v f_1(x + tv) = 1.$$

More generally, for $\tau \geq t$,

$$D_v f(x + \tau v) = D_v f_1(x + \tau v) = \frac{\tau}{t} \geq 1.$$

Thus, when $|x| \leq 1$, the value of f increases along rays emanating from x and starting on the boundary of $B_1(0)$. We have established the first assertion of the lemma and shown that the min value does not exceed $(1 - |x|)/2$. □

By the Lemma, we have reduced the overall minimization problem to that of minimizing $f_0(\xi) = |x - \xi|^2/(2t) + 1 - |\xi|$ on the closure of $B_1(0)$.

Notice that for $\xi \neq 0$,

$$Df_0(\xi) = \frac{\xi - x}{t} - \frac{\xi}{|\xi|}.$$

Thus, if $Df_0(\xi) = 0$, it must be that

$$\xi = x + t\frac{\xi}{|\xi|} \quad \text{or} \quad \left(1 - \frac{t}{|\xi|}\right)\xi = x.$$

This means $\xi/|\xi|$ is a unit vector in the direction of x. There are two possibilities:

$$\xi = x \pm t\frac{x}{|x|}.$$

The corresponding values are

$$f_0\left(x \pm t\frac{x}{|x|}\right) = \frac{t}{2} + 1 - ||x| \pm t|.$$

Thus, the critical point corresponding to $\xi = x + tx/|x|$ always gives a lower value.

Lemma 2 *There can never be a local min at $\xi = 0$.*

Proof: Let v be a unit vector and compute the directional derivative $D_v f_0(\xi)$ for $\xi \neq 0$:

$$D_v f_0(\xi) = \left(\frac{\xi - x}{t} - \frac{\xi}{|\xi|}\right) \cdot v.$$

If $x = 0$ and we take $\xi = \epsilon v$ for $\epsilon > 0$, then we find

$$D_v f_0(\epsilon v) = \left(\frac{\epsilon v}{t} - v\right) \cdot v = \frac{\epsilon}{t} - 1.$$

Notice that this quantity is negative for small ϵ and limits to -1. This means that f_0 is decreasing in every direction at $\xi = 0$, as should be expected.

If $x \neq 0$, we can take $v = x/|x|$ and $\xi = \epsilon x/|x|$ to find

$$D_v f_0(\epsilon v) = \left(\frac{\epsilon v - x}{t} - v\right) \cdot v = \frac{\epsilon - |x|}{t} - 1 \to -\frac{|x|}{t} - 1 \quad \text{as } \epsilon \searrow 0.$$

This means that f_0 is decreasing in the direction $x/|x|$ at $\xi = 0$. In either case, there can not be a local min at $\xi = 0$. $\square$

We are now ready to consider cases.

(a) $t \le \min\{|x|, 1-|x|\}$

In this case, $x + tx/|x|$ is in the closure of $B_0(0)$ and the corresponding value

$$f(x + tx/|x|) = \frac{t}{2} + 1 - ||x| + t| = 1 - |x| - \frac{t}{2}$$

is less than (or equal to) the minimum on $\partial B_1(0)$ which is

$$\frac{(1-|x|)^2}{2t}.$$

To see this, it is enough to note that taking $v = x/|x|$, we have

$$D_v f_0(v) = \left(\frac{v-x}{t} - v\right) \cdot v = \frac{1-|x|}{t} - 1 \ge 0$$

with strict inequality unless $t = 1 - |x|$. If we have strict inequality, then $\xi = x/|x|$ is not a local min. If we have equality, then $1 - |x| = t$ which means $x + tx/|x| = x/|x|$ so the values are also equal.

(b) $1 - |x| \le t \le |x|$ and $|x| \le 1$.

In this case, $x - tx/|x|$ is not a local minimum point, so there is no local min for f_0 in $B_1(0)$ and the global min is

$$f(x/|x|) = \frac{(1-|x|)^2}{2t}.$$

To see that $\xi = x - tx/|x|$ is not a local min point, let $v = x/|x|$ and compute $D_v f_0(x - tx/|x|) = -2 < 0$.

(c) $|x| \ge 1$.

The min value is $f(x) = 0$.

(d) $t \ge \max\{|x|, 1-|x|\}$ and $|x| \le 1$

In this case, the critical point for f_0 at $x + tx/|x|$ is outside $B_1(0)$. There is (possibly) another critical point at $x - tx/|x|$ inside $B_1(0)$. If $x - tx/|x| \in B_1(0)$, we claim that it is not a local min, so the global min occurs at $\xi = x/|x|$ with value

$$f(x/|x|) = \frac{(1-|x|)^2}{2t}.$$

To see this, let's compute the Hessian at $\xi = x - tx/|x|$. In fact,

$$D^2 f_0(\xi) = \left(\frac{1}{t} - \frac{1}{|\xi|}\right) I + \frac{1}{|\xi|^2} \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix}$$

Now, clearly at $\xi = x - tx/|x|$, in this case, the coefficient

$$\frac{1}{t} - \frac{1}{|\xi|} < 0.$$

This is certainly consistent with having a local max and not a local min, but we must check the effect of the non-identity matrix which becomes

$$\frac{1 + t/|x|}{(|x| + t)^3} \begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix}.$$

In fact,

$$\left\langle \begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right\langle = (x_1\eta_1 + x_2\eta_2)^2 \geq 0.$$

This looks like it could be a problem, but notice that if we take a direction orthogonal to (x_1, x_2), then this term vanishes, and the identity term gives us a negative second directional derivative. In particualar, there is no local min at $\xi = x - tx/|x|$.

(e) $|x| \leq t \leq 1 - |x|$ (and $0 \leq |x| \leq 1/2$)

As in case (a) we have

$$f(x + tx/|x|) = \frac{t}{2} + 1 - ||x| + t| = 1 - |x| - \frac{t}{2}$$

It can be checked, as in one-dimension that the minimum value $u(x,t)$ provides a piecewise smooth piecewise solution of the Hamilton-Jacobi IVP. The regularity can also be checked, and this is probably a good point to draw attention to a couple theorems in Evans' text.

Theorem 1 *(Theorem 6) If H is convex and coercive and u_0 is Lipschitz, then setting $L = H^*$, the Hopf-Lax formula*

$$u(x,t) = \min_{\xi \in \mathbb{R}^n} \left\{ tL\left(\frac{x - \xi}{t}\right) + u_0(\xi) \right\}$$

is differentiable almost everywhere and satisfies the Hamilton Jacobi IVP.

Big Lesson 1: The Hopf-Lax formula is a way to solve (at least some) Hamilton-Jacobi PDEs by solving a family of calculus problems. The min values $u(x,t)$ solve the PDE.

Note: The non-differentiability of the initial value function u_0 provided a little extra complication in our examples, but notice that the actual dependence of a minimum on parameters x and t can be complicated anyway. For example, the location of the global minimum point can change discontinuously.

Theorem 2 *(Lemma 3) If u_0 is semiconcave, then $u = u(x,t)$ given by the Hopf-Lax formula will be semi-concave.*

Theorem 3 *(Lemma 4) If H is uniformly convex, then $u = u(x,t)$ given by the Hopf-Lax formula will be semi-concave.*

Theorem 4 *(Theorem 7) If H is smooth, convex and coercive and u_0 is Lipschitz, then there can be at most one solution.*

Note: Theorem 7 does not imply the Hopf-Lax formula gives the unique solution. But if u_0 is semiconcave or H is uniformly convex, then we do know $u = u(x,t)$ given by the Hopf-Lax formula is the unique solution.

Research Tip 1: We have a pretty complete picture/theory if u_0 is semiconcave or H is uniformly convex. If one starts to relax either of these assumptions, then one quickly confronts what would probably qualify as research questions in the field of Hamilton-Jacobi equations.

Having completed the minimization problem (b) in two space dimensions, we now turn to part (a).

Restricting to vector functions $\mathbf{w} = w + \epsilon\eta$ where w is assumed to be a minimizer and $\eta \in C_c^\infty[a,b]$. Then we obtain a real valued function

$$f(\epsilon) = \mathcal{F}[w + \epsilon\eta]$$

with a minimum at $\epsilon = 0$. Calculating $f'(0) = 0$, we find

$$\int_a^b (w' \cdot \eta' - \alpha\eta_3)\, dt = 0.$$

Integrating by parts we find

$$\int_a^b (-w'' \cdot \eta - \alpha\eta_3)\, dt = 0.$$

and by the fundamental lemma of the calculus of variations

$$w_1'' = w_2'' = 0 \quad \text{and} \quad w_3'' = -\alpha.$$

Thus, w has the form

$$\begin{cases} w_1 = \xi_1(t-b) + x_1^* \\ w_2 = \xi_2(t-b) + x_2^* \\ w_3 = -\alpha(t-b)^2/2 + \xi_3(t-b) + x_3^* \end{cases}$$

Putting this w back into $\mathcal{F}$ we get a function $f = f(\xi_1, \xi_2, \xi_3)$ of three variables:

$$\begin{aligned} f(\xi) = \frac{b-a}{6} &\left[3|\xi|^2 + 6\alpha(b-a)\xi_3 + 2\alpha^2(b-a)^2\right] \\ &+ u_0(x_1^* - (b-a)\xi_1, x_2^* - (b-a)\xi_2, x_3^* - (b-a)\xi_3 - (b-a)^2/2). \end{aligned}$$

Thus, we have reduced the problem to a minimization problem for the function $f = f(\xi)$ over $\xi \in \mathbb{R}^3$.

In order to make this look a bit more like the calculus problems we have considered above, let us first set $\tilde{\xi} = w(a)$. Then, we find $f(\xi)$ becomes

... to be continued