Mean Values
for solutions of the heat equation

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The following notes are intended to address certain problems with the change of variables and other unclear points (and points simply not covered) from the lecture.

1 Heat Ball

The *heat ball* of radius r and "center" (x, t) is the set

$$E_r(x,t) = \{(\xi,\tau) : \tau \le t,\ \Phi(x-\xi, t-\tau) \ge 1/r^n\}$$

where Φ is the fundamental solution of the heat equation, so that

$$\Phi(x-\xi, t-\tau) = \frac{1}{[4\pi(t-\tau)]^{n/2}} e^{-\frac{|x-\xi|^2}{4(t-\tau)}}.$$

This last expression is always largest when $\alpha = |x - \xi| = 0$ corresponding to the "core" of the heat ball. Thus, setting the exponential term to 1, the remaining factor is increasing in τ, and we obtain a minimum value

$$\tau_{\min} = t - \frac{r^2}{4\pi}$$

corresponding to points in the ball. That is,

$$E_r(x,t) = \left\{(\xi,\tau) : \xi \in \overline{B_{\alpha(\tau)}(0)},\ t - \frac{r^2}{4\pi} \le \tau \le t\right\}$$

with

$$\alpha(\tau) = \sqrt{2n(t-\tau)\ln\left(\frac{r^2}{4\pi(t-\tau)}\right)}$$

which is obtained by solving for $\alpha = |x-\xi|$ in the equality $\Phi(x-\xi, t-\tau) = 1/r^n$. Technically, the expression for α given above applies for $t - r^2/(4\pi) \le \tau < t$. On this interval, α is seen to satisfy $\alpha'' < 0$ and the value at t is taken to be $\lim_{\tau \searrow t} \alpha(\tau) = 0$.

Notice that the "center" (x,t) is really the top center, and is on the boundary of the heat ball.

Further analysis indicates that the widest spatial radius of the heat ball is

$$\alpha(\tau_{\max}) = r\sqrt{\frac{n}{2\pi e}}$$

corresponding to

$$\tau_{\max} = t - \frac{r^2}{4\pi e}.$$

In particular, we observe that $\alpha = \alpha(\tau)$ is increasing for $t - r^2/(4\pi) \le \tau \le t - r^2/(4\pi e)$, and the restriction α_1 of α to this interval has a well defined inverse. Similarly, α restricted to $t - r^2/(4\pi e) \le \tau \le t$ is decreasing and has an inverse which will be denoted by α_0^{-1}.

2 The Mean Value Property

The main claim is that given a solution $u \in C^{2,1}(\mathcal{U} \times (0,T])$ of $u_t = \Delta u$, we have

$$u(x,t) = \frac{1}{4r^n} \int_{(\xi,\tau)\in E_r(x,t)} u(\xi,\tau) \frac{|x-\xi|^2}{(t-\tau)^2}.$$

The primary difficulty seems to be verification of the claim in the case $u \equiv 1$:

Lemma 1

$$4 = \frac{1}{r^n} \int_{(\xi,\tau)\in E_r(x,t)} \frac{|x-\xi|^2}{(t-\tau)^2}.$$

The integral on the right can be reduced (see change of variables below) to the following:

$$\int_{(\xi,\tau)\in E_1(0,0)} \frac{|\xi|^2}{\tau^2}.$$

I was unable to verify that this integral is 4, though Mathematica was able to compute it in one space dimension and said it was 4. Evans references N. Watson, Proc. London Math. Society 26 (1973) 385—417, and I've read other references, e.g., Lei Ni's *Mean Value Theorems on Manifolds* which refer to the result as "Watson's mean value formula." Watson's paper *A theory of subtemperatures in several variables,* however, does not seem to contain a proof of the result but quotes it from *A mean value theorem for the heat equation* by W. Fulks which appeared in Proc. Amer. Math. Soc. 17(i) (1966) 6—11. Unfortunately, Fulks doesn't seem to offer a direct computation of the integral in the lemma above either, but obtains it as a corollary of his general argument which is along somewhat different lines.

Assuming the lemma, Evans argument (which presumably follows an argument of Watson) is as follows:

$$\begin{aligned}\phi(r) &= \frac{1}{r^n}\int_{(\xi,\tau)\in E_r(x,t)} u(\xi,\tau)\frac{|x-\xi|^2}{(t-\tau)^2} \\ &= \frac{1}{r^n}\int_{(\xi,\tau)\in E_r(0,0)} u(\xi+x,\tau+t)\frac{|\xi|^2}{\tau^2} \\ &= \int_{(\xi,\tau)\in E_1(0,0)} u(x+r\xi,t+r^2\tau)\frac{|\xi|^2}{\tau^2}.\end{aligned}$$

Differentiating, we have

$$\begin{aligned}\phi'(r) &= \int_{(\xi,\tau)\in E_1(0,0)} Du(x+r\xi,t+r^2\tau)\cdot\xi\frac{|\xi|^2}{\tau^2} + 2r\int_{(\xi,\tau)\in E_1(0,0)} u_t(x+r\xi,t+r^2\tau)\frac{|\xi|^2}{\tau} \\ &= \frac{1}{r^n}\int_{(\xi,\tau)\in E_r(0,0)} Du(x+\xi,t+\tau)\cdot\xi\frac{|\xi|^2}{\tau^2} + \frac{2}{r^{n+1}}\int_{(\xi,\tau)\in E_r(0,0)} u_t(x+\xi,t+\tau)\frac{|\xi|^2}{\tau}.\end{aligned}$$

Next, focusing on the second term, we find

$$\begin{aligned}&\frac{2}{r^{n+1}}\int_{(\xi,\tau)\in E_r(0,0)} u_t(x+\xi,t+\tau)\frac{|\xi|^2}{\tau} \\ &\quad= \frac{4}{r^{n+1}}\int_{(\xi,\tau)\in E_r(0,0)} u_t(x+\xi,t+\tau)D\psi(\xi,\tau)\cdot\xi\end{aligned}$$

where

$$\begin{aligned}\psi(\xi,\tau) &= \ln\left[r^n\Phi(-\xi,-\tau)\right]\\ &= \ln\left[\left(\frac{r^2}{-4\pi\tau}\right)^{n/2} e^{\frac{|\xi|^2}{4\tau}}\right]\\ &= n\log r - \frac{n}{2}\log(-4\pi\tau) + \frac{|\xi|^2}{4\tau}.\end{aligned}$$

In fact,

$$D\psi(\xi,\tau) = \frac{\xi}{2\tau} \qquad \text{and} \qquad \psi\Big|_{\partial E_r(0,0)} \equiv 0. \tag{1}$$

Continuing the calculation of the second term using the divergence theorem, we find

$$\begin{aligned}&\frac{2}{r^{n+1}}\int_{(\xi,\tau)\in E_r(0,0)} u_t(x+\xi,t+\tau)\frac{|\xi|^2}{\tau}\\ &\quad= \frac{4}{r^{n+1}}\int_{(\xi,\tau)\in E_r(0,0)} \{\operatorname{div}\left[u_t(x+\xi,t+\tau)\psi(\xi,\tau)\xi\right]\\ &\qquad\qquad -\psi(\xi,\tau)Du_t(x+\xi,t+\tau)\cdot\xi - n\psi(\xi,\tau)u_t(x+\xi,t+\tau)\}\\ &\quad= -\frac{4n}{r^{n+1}}\int_{(\xi,\tau)\in E_r(0,0)} \psi(\xi,\tau)u_t(x+\xi,t+\tau)\\ &\qquad\qquad - \frac{4}{r^{n+1}}\int_{(\xi,\tau)\in E_r(0,0)} \psi(\xi,\tau)\frac{\partial}{\partial\tau}\left[Du(x+\xi,t+\tau)\cdot\xi\right].\end{aligned}$$

Considering the second integral in this expression, we can write

$$\begin{aligned}&\int_{(\xi,\tau)\in E_r(0,0)} \psi(\xi,\tau)\frac{\partial}{\partial\tau}\left[Du(x+\xi,t+\tau)\cdot\xi\right]\\ &\qquad= \int_{\xi\in B_{\alpha_{\max}}(0)}\int_{\alpha_1^{-1}(|\xi|)}^{\alpha_0^{-1}(|\xi|)} \psi(\xi,\tau)\frac{\partial}{\partial\tau}\left[Du(x+\xi,t+\tau)\cdot\xi\right]\\ &\qquad= -\int_{(\xi,\tau)\in E_r(0,0)} \psi_\tau(\xi,\tau)Du(x+\xi,t+\tau)\cdot\xi\\ &\qquad= -\int_{(\xi,\tau)\in E_r(0,0)} \left(-\frac{n}{2\tau} - \frac{|\xi|^2}{4\tau^2}\right)Du(x+\xi,t+\tau)\cdot\xi.\end{aligned}$$

Recombining expressions, we see that the second term in ϕ' now has a term which cancels the first term, and we have

$$
\begin{aligned}
\phi'(r) &= -\frac{4n}{r^{n+1}} \int_{(\xi,\tau)\in E_r(0,0)} \psi(\xi,\tau) u_t(x+\xi,t+\tau) \\
&\qquad - \frac{2n}{r^{n+1}} \int_{(\xi,\tau)\in E_r(0,0)} \frac{Du(x+\xi,t+\tau)\cdot\xi}{\tau} \\
&= -\frac{4n}{r^{n+1}} \int_{\tau_{\min}}^{0} \left[\int_{|\xi|<\alpha(\tau)} \psi(\xi,\tau) \mathrm{div} Du(x+\xi,t+\tau) \right] d\tau \\
&\qquad - \frac{2n}{r^{n+1}} \int_{(\xi,\tau)\in E_r(0,0)} \frac{Du(x+\xi,t+\tau)\cdot\xi}{\tau} \\
&= \frac{4n}{r^{n+1}} \int_{(\xi,\tau)\in E_r(0,0)} D\psi(\xi,\tau)\cdot Du(x+\xi,t+\tau) \\
&\qquad - \frac{2n}{r^{n+1}} \int_{(\xi,\tau)\in E_r(0,0)} \frac{Du(x+\xi,t+\tau)\cdot\xi}{\tau} \\
&= 0
\end{aligned}
$$

in view of (1).

This establishes, at least, that ϕ is a constant. The constant is then easily seen to be

$$
\phi(r) \equiv \lim_{r\searrow 0} \int_{(\xi,\tau)\in E_1(0,0)} u(x+r\xi,t+r^2\tau)\frac{|\xi|^2}{\tau^2} = u(x,t) \int_{(\xi,\tau)\in E_1(0,0)} \frac{|\xi|^2}{\tau^2},
$$

and if we had a verification that the integral appearing above has value 4, then we would be done, since the expression appearing in the main result is $\phi(r)/4$.