Green's Function for a Domain

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In view of the importance of football at Georgia Tech, I have decided to compose the following discussion as a follow-up to Tuesday's lecture. There *were* a couple sign errors in my notes. Hopefully, I've got it right below, but you should check it.

Recall that our discussion started with Green's second identity involving two C^2 functions on a domain Ω:

$$\int_\Omega (u\Delta v - v\Delta u) = \int_{\partial\Omega} (uDv - vDu) \cdot n$$

where n is the outward unit normal to $\partial\Omega$ and everything is smooth enough for this to make sense, e.g., $u, v \in C^2(\bar{\Omega})$ and $\partial\Omega \in C^1$. You should be able to derive this easily from the Gauss' divergence theorem.

As a first step take $v(\xi) = \Phi(x - \xi)$ for $x \in \mathcal{U}$ fixed where $\mathcal{U}$ is an open bounded C^1 domain and Φ is a fundamental solution tending to $+\infty$ at the singularity.[1] Taking, furthermore, $\Omega = \Omega_\epsilon = \mathcal{U} \backslash B_\epsilon(x)$ with $B_\epsilon(x) \subset\subset \mathcal{U}$, we note that

$$Dv(\xi) = -D\Phi(x - \xi) \qquad \text{and} \qquad \Delta v(\xi) = \Delta\Phi(x - \xi) = 0 \quad \text{on } \Omega.$$

Therefore, Green's second identity reads

$$\begin{aligned} -\int_{\xi \in \Omega_\epsilon} \Phi(x - \xi)\Delta u(\xi) = &\int_{\xi \in \partial\mathcal{U}} [-u(\xi)D\Phi(x - \xi) - \Phi(x - \xi)Du(\xi)] \cdot n \\ &+ \int_{\partial B_\epsilon(x)} [u(\xi)D\Phi(x - \xi) + \Phi(x - \xi)Du(\xi)] \cdot \nu, \end{aligned} \tag{1}$$

[1] The authoritative text on the subject of Laplace's equation by Gilbarg and Trudinger uses $\Gamma = -\Phi$ as the standard fundamental solution.

where n is the outward unit normal to $\partial\mathcal{U}$ and ν is the outward unit normal to $\partial B_\epsilon(x)$. In view of the fact that

$$\Phi(x) = \begin{cases} -\ln|x|/(2\pi), & n = 2, \\ 1/[n\omega_n(n-2)|x|^{n-2}], & n \geq 3, \end{cases}$$

and

$$D\Phi(x) = -\frac{1}{n\omega_n}\frac{x}{|x|^n}$$

where $\omega_n = |B_1(0)|$ is the n-dimensional measure of the unit ball in $\mathbb{R}^n$, we see

$$\begin{aligned} \left|\int_{\xi\in\partial B_\epsilon(x)} \Phi(x-\xi)Du(\xi)\cdot\nu\right| &\leq |Du|_{L^\infty(\overline{B_\epsilon(x)})}\int_{\xi\in\partial B_\epsilon(x)} |\Phi(x-\xi)| \\ &\leq |Du|_{L^\infty(\overline{B_\epsilon(x)})}\int_{\xi\in\partial B_\epsilon(x)} \ln\epsilon/(n\omega_n\epsilon^{n-2}) \\ &= |Du|_{L^\infty(\overline{B_\epsilon(x)})}\epsilon\ln\epsilon \\ &\to 0 \quad \text{as } \epsilon\to 0, \end{aligned}$$

and

$$\begin{aligned} \int_{\xi\in\partial B_\epsilon(x)} u(\xi)D\Phi(x-\xi)\cdot\nu &= \int_{\xi\in\partial B_\epsilon(x)} u(\xi)\left(-\frac{x-\xi}{n\omega_n|x-\xi|^n}\right)\cdot\frac{\xi-x}{|\xi-x|} \\ &= \frac{1}{n\omega_n\epsilon^{n-1}}\int_{\xi\in\partial B_\epsilon(x)} u(\xi) \\ &= \frac{1}{|\partial B_\epsilon(x)|}\int_{\xi\in\partial B_\epsilon(x)} u(\xi) \\ &\to u(x) \quad \text{as } \epsilon\to 0. \end{aligned}$$

Therefore, taking the limit in (1), we find

$$\begin{aligned} -\int_{\xi\in\mathcal{U}} \Phi(x-\xi)\Delta u(\xi) = &\int_{\xi\in\partial\mathcal{U}} [-u(\xi)D\Phi(x-\xi) - \Phi(x-\xi)Du(\xi)]\cdot n \\ &+ u(x). \end{aligned}$$

That is,

$$\boxed{\begin{aligned} u(x) = -&\int_{\xi\in\mathcal{U}} \Phi(x-\xi)\Delta u(\xi) \\ &\int_{\xi\in\partial\mathcal{U}} [u(\xi)D\Phi(x-\xi) + \Phi(x-\xi)Du(\xi)]\cdot n \end{aligned}}$$

This is called *Green's Representation Formula.* It does not quite give a formula for the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{on } \mathcal{U} \\ u\big|_{\partial\mathcal{U}} = u_0 \end{cases}$$

because the *Neumann data* $Du \cdot n$ is not specified by the problem on $\partial\mathcal{U}$.

In order to deal with this defect, we start by writing down Green's second identity again for u and any $w \in C^2(\mathcal{U})$ with $\Delta w = 0$:

$$\int_{\mathcal{U}} w\Delta u = \int_{\partial\mathcal{U}} [wDu - uDw]\cdot n.$$

Now, if for an arbitrary fixed $x \in \mathcal{U}$, we can further arrange that w is a solution of the specific Dirchlet problem

$$\begin{cases} \Delta w = 0 & \text{on } \mathcal{U} \\ w(\xi)\big|_{\xi\in\partial\mathcal{U}} = \Phi(x-\xi)\big|_{\xi\in\partial\mathcal{U}}, \end{cases}$$

then

$$\int_{\partial\mathcal{U}} wDu\cdot n = \int_{\xi\in\partial\mathcal{U}} \Phi(x-\xi)Du(\xi)\cdot n,$$

and we can replace the integral with the Neumann data for u with a sum of integrals involving only w. In fact, Green's representation formula can be written as

$$\begin{aligned}
u(x) = -\int_{\xi\in\mathcal{U}} &\Phi(x-\xi)\Delta u(\xi) \\
&+ \int_{\xi\in\partial\mathcal{U}} u(\xi) D\Phi(x-\xi)\cdot n \\
&\qquad + \int_{\mathcal{U}} w\Delta u + \int_{\partial\mathcal{U}} uDw\cdot n \\
= -\int_{\xi\in\mathcal{U}} &[\Phi(x-\xi) - w(\xi)]\Delta u(\xi) \\
&+ \int_{\xi\in\partial\mathcal{U}} u(\xi)[D\Phi(x-\xi) + Dw(\xi)]\cdot n.
\end{aligned}$$

On the other hand, setting $G^+(x,\xi) = \Phi(x-\xi) - w(\xi;x)$, we find

$$D^\xi G^+(x,\xi) = -D\Phi(x-\xi) - Dw(\xi)$$

where we have used the notation D^ξ to explicitly denote a gradient in the ξ variables. (Note this is different from the other two notations we have used, namely, D^j where j is a positive integer to denote the total array of derivatives of order j or D^α where α is a multi-index to denote a particular partial derivative.) Making this final substitution, we find

$$\boxed{u(x) = -\int_{\xi\in\mathcal{U}} G^+(x,\xi)\Delta u(\xi) - \int_{\xi\in\partial\mathcal{U}} u(\xi) D^\xi G^+(x,\xi)\cdot n.}$$

We remark explicitly that *Green's function* for the domain $\mathcal{U}$ is defined (by me) to be

$$G^+(x,\xi) = \Phi(x-\xi) - w(\xi;x).$$

The notation D^ξ appearing in the formula above turns out to be superfluous due to the fact that the Green's function is symmetric:

$$G^+(x,\xi) = G^+(\xi,x).$$

I will leave it to you to double check (the signs of) the argument for this we gave in class.

We finish this discussion with a compilation of the varying definitions and versions of this same topic given by Evans and Gilbarg and Trudinger. It may be a useful exercise for you to harmonize the various accounts and make sure all the formulas come to the same thing.

1 Evans' and Gilbarg and Trudinger's Green's Function

The main difference between Evans' treatment and the one above is that he uses $\xi - x$ for the argument of Φ rather than $x - \xi$ which we have chosen to be consistent with the expression appearing in convolution integrals. Thus, his version of Green's representation formula is

$$u(x) = \int_{\xi \in \partial \mathcal{U}} \left[\Phi(\xi - x) \frac{\partial u}{\partial n}(\xi) - u(\xi) \frac{\partial \Phi}{\partial n}(\xi - x) \right] - \int_{\xi \in \mathcal{U}} \Phi(\xi - x) \Delta u(\xi).$$

Gilbarg and Trudinger (G-T) write Poisson's equation as $\Delta u = f$. (As a notational device simply to distinguish their treatment in this context, I will also use their name for the domain of u—which is almost universal in the literature. They call the domain Ω. I do not know why Evans does not.) Green's representation formula from G-T is

$$u(x) = \int_{\xi \in \partial \Omega} \left[u(\xi) \frac{\partial \Gamma}{\partial n}(\xi - x) - \Gamma(\xi - x) \frac{\partial u}{\partial n}(\xi) \right] + \int_{\xi \in \Omega} \Gamma(\xi - x) \Delta u(\xi).$$

These are obviously very easy to translate. Evans writes

$$G(x, \xi) = \Phi(\xi - x) - \phi(\xi; x)$$

where $\phi = \phi(\xi) = \phi(\xi; x)$ solves

$$\begin{cases} \Delta \phi = 0 \quad \text{on } \mathcal{U} \\ \phi(\xi)\big|_{\xi \in \partial \mathcal{U}} = \Phi(\xi - x)\big|_{\xi \in \partial \mathcal{U}}, \end{cases}$$

while G-T has

$$G^{-}(x, \xi) = \Gamma(\xi - x) - h(\xi; x)$$

where

$$\begin{cases} \Delta h = 0 \quad \text{on } \Omega \\ h(\xi)\big|_{\xi \in \partial \Omega} = \Gamma(x - \xi)\big|_{\xi \in \partial \Omega}. \end{cases}$$

And finally, Evans has

$$u(x) = -\int_{\xi\in\mathcal{U}} G(x,\xi)\Delta u(\xi) - \int_{\xi\in\partial\mathcal{U}} u(\xi)\frac{\partial G}{\partial n}(x,\xi).$$

This is essentially the same as mine. Does this make sense, or is there still a sign error somewhere? Are G and G^+ the same function? On the other hand G-T has

$$u(x) = \int_{\xi\in\Omega} G^-(x,\xi)\Delta u(\xi) + \int_{\xi\in\partial\Omega} u(\xi)\frac{\partial G^-}{\partial n}(x,\xi).$$

This makes sense of course, because they are writing down a solution of $\Delta u = f$.

Finally, substituting the potentials and boundary values, we get:
McCuan's solution:

$$u(x) = \int_{\xi\in\mathcal{U}} G^+(x,\xi)f(\xi) - \int_{\xi\in\partial\mathcal{U}} u_0(\xi)D^\xi G^+(x,\xi)\cdot n.$$

and Evans' solution:

$$u(x) = \int_{\xi\in\mathcal{U}} G(x,\xi)f(\xi) - \int_{\xi\in\partial\mathcal{U}} u_0(\xi)\frac{\partial G}{\partial n}(x,\xi)$$

for

$$\begin{cases} -\Delta u = f & \text{on } \mathcal{U} \\ u\big|_{\partial\mathcal{U}} = u_0 \end{cases}$$

And we have Gilbarg and Trudinger's solution:

$$u(x) = \int_{\xi\in\Omega} G^-(x,\xi)f(\xi) + \int_{\xi\in\partial\Omega} u_0(\xi)\frac{\partial G^-}{\partial n}(x,\xi)$$

for the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ u\big|_{\partial\Omega} = u_0 \end{cases}$$

How is G^- related to G and G^+?