Geometric PDE

and

The Magic of Maximum Principles

...

Alexandrov's Theorem

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Outline

1. Young's PDE

2. Geometric Interpretation

3. Maximum and Comparison Principles

4. Alexandrov's Theorem

Variation of Area

$$\mathcal{G}[u] = \int_\Omega \sqrt{1 + |Du|^2}$$

(Nonparametric) Volume Constrained Plateau Problem:

(P):

Minimize $\mathcal{G}$ *over*

$$\mathcal{A} = \{u \in C^2(\Omega) \cap C^0(\bar{\Omega}) : u\Big|_{\partial\Omega} = u_0,\ \int_\Omega u = V_0\}.$$

Lagrange Multiplier Problem

(P)$_\lambda$:

Minimize

$$\tilde{\mathcal{G}}[u] = \int_\Omega \sqrt{1 + |Du|^2} - \lambda \int_\Omega u$$

over

$$\tilde{\mathcal{A}} = \{u \in C^2(\Omega) \cap C^0(\bar{\Omega}) : u\Big|_{\partial\Omega} = u_0\}.$$

Lagrange Lemma

If u solves $(\mathsf{P})_\lambda$, and it happens that $\int_\Omega u = V_0$, then u solves Problem (P).

Proof:

Euler-Lagrange PDE

$$\delta\tilde{\mathcal{G}}[\eta] = \int_\Omega \left(\frac{Du}{\sqrt{1+|Du|^2}} \cdot D\eta - \lambda\eta \right)$$

Theorem: If u solves $(\mathsf{P})_\lambda$, then

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}} \right) = -\lambda.$$

Constrained Plateau BVP:

$$\begin{cases} \text{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = -\lambda \text{ on } \Omega \\ \\ u\Big|_{\partial\Omega} = u_0 \end{cases}$$

Young Operator:

$$\mathcal{M}u = \text{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$$
$$= \frac{1}{(1+|Du|^2)^{3/2}}\left[(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy}\right]$$

Quasilinear Elliptic Operator(s)

Linear Partial Differential Operator:

$$Lu = \sum a_{ij}(x) D_{ij} u + \sum b_j(x) D_j u + c(x) u$$

Quasilinear:

$$Qu = \sum A_{ij}(x, y, Du) D_{ij} u + C(x, u, Du)$$

Examples: Laplace

$$a_{ij} = \delta_{ij}$$

Young

$$(A_{ij}) = \frac{1}{(1 + |Du|^2)^{3/2}} \begin{pmatrix} 1 + u_y^2 & -u_x u_y \\ -u_x u_y & 1 + u_x^2 \end{pmatrix}$$

Ellipticity

$\exists\ \lambda_0 > 0$ such that

$$\sum a_{ij}(x)\xi_i\xi_j \geq \lambda_0|\xi|^2$$

(This is called uniform ellipticity.)

Young's Operator on $\mathbb{R}^2$:

$$\frac{1}{(1+|Du|^2)^{3/2}}\left[(1+u_y^2)\xi_1^2 - 2u_xu_y\xi_1\xi_2 + (1+u_x^2)\xi_2^2\right] = \frac{1}{(1+|Du|^2)^{3/2}}\left[|\right.$$

Theorem: $\mathcal{M}$ is uniformly elliptic if there is some $M > 0$ such that $|Du| < M$ on Ω. (Bounded Gradient Condition)

Geometric Interpretations

a. Solutions minimize area with respect to a volume constraint.

They “are” pieces of soap bubbles or surfaces bounding liquids in equilibrium in zero gravity.

b. Mean Curvature: The Young operator is also called the

mean curvature operator

Surface Geometry Calculation

$$\mathbf{v}_0 \in T_{\mathbf{x}_0}\mathbb{R}^2 \longleftrightarrow \mathbf{w}_0 \in T_p\mathcal{S}$$

$$\alpha(t) = (\mathbf{x}_0 + \mathbf{v}_0 t, u(\mathbf{x}_0 + \mathbf{v}_0 t))$$

$$\mathbf{w}_0 = \alpha'(0) = (a, b, au_x + bu_y)$$

Arclength

$$s = \int_0^t |\alpha'(\tau)| d\tau$$

$$\frac{dt}{ds} = \frac{1}{|\alpha'|} = \frac{1}{\sqrt{a^2 + b^2 + (au_x + bu_y)^2}}$$

(re)Parameterization by arclength: $\gamma(s) = \alpha(t(s))$

$$\dot{\gamma} = \frac{\alpha'}{|\alpha'|}$$

Curvature Vector

$$\vec{k} = \frac{d^2\gamma}{ds^2} = \left[\frac{\alpha''}{|\alpha'|} - \frac{\alpha'}{|\alpha'|^3}(\alpha' \cdot \alpha'')\right]\frac{1}{|\alpha'|}$$

Applicable for any curve in $\mathbb{R}^3$

Normal Curvature

For a curve on a surface, the *normal curvature* is

$$k = \vec{k} \cdot N = \frac{N \cdot \alpha''}{|\alpha'|^2}$$

Musnier's Theorem: All curves with the same direction in $T_p\mathcal{S}$ have the same normal curvature.

Calculation

$$\alpha' = (a, b, au_x(\mathbf{x}_0 + \mathbf{v}_0 t) + bu_y(\mathbf{x}_0 + \mathbf{v}_0 t))$$

$$\alpha'' = (0, 0, a^2 u_{xx} + 2ab u_{xy} + b^2 u_{yy})$$

$$N = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |Du|^2}}$$

Normal Curvature:

$$k = \frac{1}{\sqrt{1 + |Du|^2}} \frac{a^2 u_{xx} + 2ab u_{xy} + b^2 u_{yy}}{a^2 + b^2 + (au_x + bu_y)^2}$$

Next Task

Find the normal curvature associated with a direction $\mathbf{w}$ orthogonal to $\mathbf{w}_0$ in $T_p\mathcal{S}$

Basis for $T_p\mathcal{S}$: $\{(1, 0, u_x), (0, 1, u_y)\}$

ONB: $\mathbf{u}_1 = (1, 0, u_x)/\sqrt{1 + u_x^2}$

$$\mathbf{u}_2 = \frac{(-u_x u_y, 1 + u_x^2, u_y)}{\sqrt{(1 + u_x^2)(1 + u_x^2 + u_y^2)}}$$

Components of $\mathbf{w}_0 = (a, b, au_x + bu_y)$:

$$\mathbf{w}_0 \cdot \mathbf{u}_1 = \frac{a + au_x^2 + bu_x u_y}{\sqrt{1 + u_x^2}}$$

$$\mathbf{w}_0 \cdot \mathbf{u}_2 = \frac{b\sqrt{1 + u_x^2 + u_y^2}}{\sqrt{1 + u_x^2}}$$

$$\mathbf{w} = \mathbf{w}_0^{\perp} = -(\mathbf{w}_0 \cdot \mathbf{u}_2)\mathbf{u}_1 + (\mathbf{w}_0 \cdot \mathbf{u}_1)\mathbf{u}_2$$

$$\mathbf{w}_0 \cdot \mathbf{u}_1 = \frac{a + a u_x^2 + b u_x u_y}{\sqrt{1 + u_x^2}}, \qquad \mathbf{w}_0 \cdot \mathbf{u}_2 = \frac{b\sqrt{1 + u_x^2 + u_y^2}}{\sqrt{1 + u_x^2}}$$

$$\mathbf{u}_1 = \frac{(1, 0, u_x)}{\sqrt{1 + u_x^2}}, \qquad \mathbf{u}_2 = \frac{(-u_x u_y, 1 + u_x^2, u_y)}{\sqrt{(1 + u_x^2)(1 + u_x^2 + u_y^2)}}$$

$$\tilde{a} = -\frac{b\sqrt{1+u_x^2+u_y^2}}{\sqrt{1+u_x^2}}\frac{1}{\sqrt{1+u_x^2}} - \frac{u_x u_y(a+au_x^2+bu_x u_y)}{(1+u_x^2)\sqrt{1+u_x^2+u_y^2}}$$

$$= -\frac{b+bu_y^2+au_x u_y}{\sqrt{1+u_x^2+u_y^2}}$$

$$\tilde{b} = \frac{a+au_x^2+bu_x u_y}{\sqrt{1+u_x^2+u_y^2}}$$

Notation

$$W = 1 + u_x^2 + u_y^2, \quad R = bu_x - au_y, \quad p = u_x, \quad q = u_y$$

$$\tilde{a} = -\frac{bW - pR}{\sqrt{W}} \qquad \tilde{b} = \frac{aW + qR}{\sqrt{W}}$$

Orthogonal Normal Curvature

$$\tilde{k} = \frac{1}{\sqrt{1+|Du|^2}} \frac{\tilde{a}^2 u_{xx} + 2\tilde{a}\tilde{b} u_{xy} + \tilde{b}^2 u_{yy}}{\tilde{a}^2 + \tilde{b}^2 + (\tilde{a} u_x + \tilde{b} u_y)^2}$$

We know $|\mathbf{w}| = |\mathbf{w}_0^{\perp}| = |\mathbf{w}_0|$. So

$$\tilde{a}^2 + \tilde{b}^2 + (\tilde{a} u_x + \tilde{b} u_y)^2 = a^2 + b^2 + (a u_x + b u_y)^2$$

(Caution: This is tricky to check directly!)

Sum of Orthogonal Normal Curvatures:

$$k + \tilde{k} = \frac{1}{\sqrt{1 + |Du|^2}} \frac{(a^2 + \tilde{a}^2)u_{xx} + 2(ab + \tilde{a}\tilde{b})u_{xy} + (b^2 + \tilde{b}^2)u_{yy}}{a^2 + b^2 + (au_x + bu_y)^2}$$

Check/Simplify:

$$\tilde{a}^2 = \frac{b^2W^2 - 2bpWR + p^2R^2}{W}$$

$$\tilde{a}\tilde{b} = -\frac{abW^2 + (bq - ap)WR - pqR^2}{W}$$

$$\tilde{b}^2 = \frac{a^2W^2 - 2aqWR + q^2R^2}{W}$$

$$k + \tilde{k} = \frac{1}{\sqrt{1 + |Du|^2}} \frac{(a^2 + \tilde{a}^2)u_{xx} + 2(ab + \tilde{a}\tilde{b})u_{xy} + (b^2 + \tilde{b}^2)u_{yy}}{a^2 + b^2 + (au_x + bu_y)^2}$$

Check/Simplify:

$$
\begin{aligned}
a^2 + \tilde{a}^2 &= \frac{1}{W}(a^2 W + b^2 W^2 - 2bpWR + p^2 R^2) \\
&= \frac{1}{W}[a^2 W + b^2 W^2 - 2pW(b^2 p - abq) + p^2(b^2 p^2 - 2abpq + a^2 q^2)] \\
&= \frac{1}{W}[a^2(W + p^2 q^2) + b^2(W^2 - 2p^2 W + p^4) + 2abpq(1 + q^2)] \\
&= \frac{1}{W}[a^2(1 + p^2)(1 + q^2) + b^2(1 + q^2)^2 + 2abpq(1 + q^2)] \\
&= \frac{1 + q^2}{W}[a^2(1 + p^2) + b^2(1 + q^2) + 2abpq] \\
&= \frac{1 + q^2}{W}[a^2 + b^2 + (ap + bq)^2]
\end{aligned}
$$

Conclusion (Geometric Interpretation)

$$k + \tilde{k} = \frac{1}{(1 + |Du|^2)^{3/2}} \left[(1 + u_y^2) u_{xx} + \cdots \right]$$

Check:

$$ab + \tilde{a}\tilde{b} = -\frac{pq}{W}[a^2 + b^2 + (ap + bq)^2] \quad \tilde{b}^2 = \frac{1 + p^2}{W}[a^2 + b^2 + (ap + bq)^2]$$

Then:

$$k + \tilde{k} = \mathcal{M}u$$

$$H = \frac{k + \tilde{k}}{2} \qquad \text{Mean Curvature}$$

3. Maximum and Comparison Principles

0. Weak Maximum Principle

Assumptions: $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$

Ω open, bounded

$Lu \geq 0$ with $Lu = \sum a_{ij} D_{ij} u + \sum b_j D_j u$ (Note: $c = 0$)

uniformly elliptic

Conclusion:

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$$

Proof in case of strict inequality:

Assume $Lu > 0$ on Ω. Then

$$u(x) < \max_{\bar{\Omega}} u \text{ for all } x \in \Omega$$

IDEA: At an interior max:

$$Du(\mathbf{x}_0) = 0 \qquad D^2u(\mathbf{x}_0) \leq 0$$

$$\Longrightarrow \qquad \sum a_{ij}(\mathbf{x}_0)\, D_{ij}u(\mathbf{x}_0)\ > 0$$

Details:

$$Lu(\mathbf{x}_0) = \text{trace}[(aij)D^2u]$$

Second condition: $D^2u(\mathbf{x}_0) \leq 0$

Multivariable Taylor's Formula:

$$u(x) = u(\mathbf{x}_0) + Du(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}\langle D^2u(\mathbf{x}_*)(\mathbf{x} - \mathbf{x}_0), \mathbf{x} - \mathbf{x}_0\rangle$$

Take $\mathbf{v} = (\mathbf{x} - \mathbf{x}_0)/|\mathbf{x} - \mathbf{x}_0|$ fixed. Then at max

$$\left\langle D^2u(\mathbf{x}_*)\frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}, \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}\right\rangle = \frac{2[u(\mathbf{x}) - u(\mathbf{x}_0)]}{|\mathbf{x} - \mathbf{x}_0|^2}$$

$$\Longrightarrow \qquad \langle D^2u(\mathbf{x}_*)\mathbf{v}, \mathbf{v}\rangle \leq 0$$

(Such a matrix is *negative semi-definite*.)

Getting the Contradiction (more details)

$A = (a_{ij})$ (positive definite) $\quad H = (b_{ij})$ (negative semi – definite)

(both symmetric)

Diagonalize H:

$$UHU^{-1} = \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

U orthogonal and $\lambda_1, \ldots, \lambda_n \leq 0$

Getting the contradiction (continued)

$$\begin{aligned}\text{trace}(AH) &= \text{trace}(AHU^{-1}U)\\ &= \text{trace}(UAHU^{-1})\\ &= \text{trace}(UAU^{-1}\Lambda)\end{aligned}$$

Check that $M = UAU^{-1}$ has positive diagonal entries.

$$\text{trace}(M\Lambda) = \sum m_{jj}\lambda_j \le 0. \qquad \square$$

1. E. Hopf Boundary Point Lemma

Assumptions: $u \in C^2(\Omega_0) \cap C^0(\bar{\Omega}_0)$, $\Omega_0 = B_r(\xi_0)$,

$Lu \geq 0$, L uniformly elliptic, $c = 0$

Know:

$$\max_{\bar{\Omega}_0} u = \max_{\partial\Omega_0} u$$

We can take $\mathbf{x}_0 \in \partial B_r(\xi_0)$ with $u(\mathbf{x}_0) = \max u$

Conclusion: *If* $u(x) < u(\mathbf{x}_0)$ for $x \in \Omega_0$, then

$$\overline{\lim_{h \searrow 0}} \; \frac{u(\mathbf{x}_0) - u(\mathbf{x}_0 - h(\mathbf{x}_0 - \xi_0)/r)}{h} > 0$$

2. E. Hopf Strong Maximum Principle

Assumptions: $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$,

Ω open, bounded

$Lu \geq 0$, L uniformly elliptic, $c = 0$

Conclusion: Either $u \equiv \max u$ or then

$$u(x) < \max_{\partial\Omega} u \qquad \forall x \in \Omega$$

3. Comparison Principle

Assumptions: $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$, (alt. $u, v \in C^2(\bar{\Omega})$)

Ω open, bounded

$\mathcal{M}u \geq \mathcal{M}v$, $\mathcal{M}u = \sum A_{ij}(Du)D_{ij}u$ uniformly elliptic

Conclusion: If $u\Big|_{\partial\Omega} \leq v\Big|_{\partial\Omega}$, then $u \equiv v$ or

$$u(x) < v(x) \qquad \forall x \in \Omega$$

Proof:

$$
\begin{aligned}
L(u-v) &= \mathcal{M}u - \mathcal{M}v \\
&= \sum A_{ij}(Du)D_{ij}u - \sum A_{ij}(Dv)D_{ij}v \\
&= \sum A_{ij}((1-t)Dv + tDu)[(1-t)D_{ij}v + tD_{ij}u]\Big|_{t=0}^{1} \\
&= \sum \int_0^1 \frac{d}{dt}\left\{A_{ij}((1-t)Dv + tDu)[(1-t)D_{ij}v + tD_{ij}u]\right\}\, dt \\
&= \sum \left(\int_0^1 A_{ij}((1-t)Dv + tDu)\, dt\right) D_{ij}(u-v) \\
&+ \sum_{i,j,k} \left(\int_0^1 \frac{\partial A_{ij}}{\partial p_k}((1-t)Dv + tDu)[(1-t)D_{ij}v + tD_{ij}u]\, dt\right) D_k(u -
\end{aligned}
$$

$$
= \sum a_{ij}(x)D_{ij}(u-v) + \sum_k b_k(x)D_k(u-v)
$$

where

$$a_{ij} = \int_0^1 A_{ij}((1-t)Dv + tDu)\,dt$$

$$b_k = \sum_{i,j} \int_0^1 \frac{\partial A_{ij}}{\partial p_k}((1-t)Dv + tDu)[(1-t)D_{ij}v + tD_{ij}u]\,dt$$

We need a_{ij}, b_k continuous (and bounded) in $\bar{\Omega}$. (alt. $u, v \in C^2(\bar{\Omega})$)

Rest of the proof:

$w = u - v$

$Lw \geq 0$

$w\Big|_{\partial\Omega} \leq 0$

Strong Maximum Principle $\Longrightarrow$ $w \equiv \max w$ or

$$w(x) < \max w \leq 0 \quad \text{for } x \in \Omega \qquad \square$$

4. Boundary Point Comparison Principle

Assumptions: $u, v \in C^2(\bar{\Omega}_0)$,

$\Omega_0 = B_r(\xi_0)$

$\mathcal{M}u \geq \mathcal{M}v$, $\mathcal{M}u = \sum A_{ij}(Du)D_{ij}u$ uniformly elliptic

Conclusion: If $u\Big|_{\partial\Omega} \leq v\Big|_{\partial\Omega}$ with equality at $\mathbf{x}_0 \in \partial\Omega_0$, then $u \equiv v$ or

$$\overline{\lim_{h \searrow 0}} \frac{v(\mathbf{x}_0 - h(\mathbf{x}_0 - \xi_0)/r) - u(\mathbf{x}_0 - h(\mathbf{x}_0 - \xi_0)/r)}{h} > 0$$

Alexandrov's Theorem

Assumptions: $\mathcal{S}$ is a parametric surface

compact, embedded, no boundary

CMC (constant mean curvature)

Conclusion: $\mathcal{S}$ is a (round) sphere.

Reflection Lemma

Given any direction $\mathbf{v} \in \mathbb{R}^3$ we can show that $\mathcal{S}$ has a plane $\Pi_{\mathbf{v}}$ of reflective symmetry orthogonal to $\mathbf{v}$.

Moreover, $\Pi_{\mathbf{v}}$ divides $\mathcal{S}$ into two graphs of locally bounded gradient over $\Pi_{\mathbf{v}}$.

Proof:

$\mathcal{S}$ encloses a volume $\mathcal{V} \subset \mathbb{R}^3$.

$$\mathcal{S} = \partial \mathcal{V}$$

Move Π up and reflect. Initially

(C):

> *$\hat{\mathcal{S}}$ is a graph of bounded gradient, and*
>
> *$\hat{\mathcal{S}} \subset \mathcal{V}$.*

$$t_* = \sup\{t : (C) \text{ holds}\}$$

Proof of Alexandrov's Theorem (continued)

There are two ways condition (C) can cease to hold

Interior Touching:

$$\mathcal{M}u = \mathcal{M}v \qquad u \leq v$$

Comparison Principle $\Longrightarrow u \equiv v$

Conclusion: $\{p \in \hat{\mathcal{S}} \cap \mathcal{S}\}$ is open (and closed) in $\hat{\mathcal{S}}$.

Proof of Alexandrov's Theorem (continued)

Second way (C) can cease to hold

Boundary Touching: $\mathcal{M}u = \mathcal{M}v$ on a half neighborhood.

$u \leq v$

$Du(\mathbf{x}_0) = Dv(\mathbf{x}_0)$

Boundary Comparison Principle $\Longrightarrow u \equiv v$ $\square$

Comments on other details

Have a good break!

See you in the Spring.