Main existence and uniqueness theorem

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Given

$$Lu = -\sum D_i(a_{ij}Dju) + \sum_j b_j D_j u + cu$$

with the coeficients a_{ij}, b_j, c bounded and measurable, we are looking for $u \in H_0^1(\mathcal{U})$ such that

$$B(u,v) = \langle u, v\rangle_{L^2} \qquad \text{for all } v \in H_0^1(\mathcal{U})$$

where

$$B(u,v) = \int_\Omega \sum_{i,j} a_{ij} D_j u D_i v + \int_\Omega \sum_j b_j v D_j u + \int_\Omega cuv.$$

is the associated bilinear form. Let us take as our starting point Evans' first existence theorem:

Theorem 1 *If L is (uniformly) elliptic, then there is some M such that*

$$\tilde{B}(u,v) = B(u,v) + \mu\langle u, v\rangle_{L^2}$$

is bounded and coercive for all $\mu \geq M$. Therefore, by the Lax-Milgram theorem, there is a unique $u \in H_0^1(\mathcal{U})$ for each $f \in L^2(\mathcal{U})$ such that

$$\tilde{B}(u,v) = B(u,v) + \mu\langle u, v\rangle_{L^2} = \langle f, v\rangle_{L^2} \qquad \text{for all } v \in H_0^1(\mathcal{U}).$$

Denoting the solution operator by $\Lambda = \Lambda_\mu : L^2 \to H_0^1$ and setting $\tilde{\Lambda} = I \circ \Lambda : L^2 \to L^2$ where I is the natural compact embedding of $H_0^1(\mathcal{U})$ into $L^2(\mathcal{U})$, we have shown the following:

Lemma 1 *(solution operator) $\Lambda : L^2 \to H_0^1$ is a bounded linear operator and, consequently, $\tilde{\Lambda} : L^2 \to L^2$ is a compact operator.*

Our objective here is to prove the following:

Theorem 2 *(main existence/uniqueness theorem) If L is (uniformly) elliptic and $c \geq 0$, then for each $f \in L^2(\mathcal{U})$, there is a unique $u \in H_0^1(\mathcal{U})$ such that*

$$B(u,v) = \langle f, v\rangle_{L^2} \qquad \text{for all } v \in H_0^1(\mathcal{U}).$$

Proof: Let $\mu = \mu_0$ be fixed with $\mu_0 > M$ as in Evans' first existence theorem so that the solution embedding $\tilde{\Lambda} : L^2 \to L^2$ is compact. We can (and will) also assume $\mu_0 > 0$.

Note that a function $u \in H_0^1(\mathcal{U})$ satisfies

$$B(u,v) = \langle f, v\rangle_{L^2} \qquad \text{for all } v \in H_0^1(\mathcal{U}).$$

if and only if

$$\tilde{B}(u,v) - \mu_0 \langle u, v\rangle_{L^2} = \langle f, v\rangle_{L^2} \qquad \text{for all } v \in H_0^1(\mathcal{U}),$$

i.e., if and only if $u = \mu_0 \Lambda u + \Lambda f$, i.e., if and only if

$$\tilde{\Lambda} u - \frac{1}{\mu_0} I u = -\frac{1}{\mu_0} \tilde{\Lambda} f.$$

Here we have applied the natural embedding $I : H_0^1 \to L^2$ to both sides. Notice that we can extend I to L^2 and consider also the operator

$$\tilde{\Lambda} - \frac{1}{\mu_0} \bar{I} : L^2 \to L^2$$

where $\bar{I} : L^2 \to L^2$ is the trivial/identity extension operator. Of course, this operator will agree with $\tilde{\Lambda} - I/\mu_0$ on the subspace $H_0^1(\mathcal{U})$.

By the Fredholm alternative,[1] either

(i) $\lambda = -1/\mu_0$ is an eigenvalue for $\tilde{\Lambda}$, or

(ii) For Each $\tilde{f} \in L^2$, there is a unique $u \in L^2$ for which

$$\left(\tilde{\Lambda} - \frac{1}{\mu_0} \bar{I} \right) u = \tilde{f}.$$

[1] See the auxiliary resuts at the end.

If (i) were to hold, then we would get a *nonzero* function $u \in L^2$ for which

$$\tilde{\Lambda}u - \frac{1}{\mu_0}\bar{I}u = 0.$$

Since we know in fact that $\tilde{\Lambda}(L^2(\mathcal{U})) \subset H_0^1(\mathcal{U})$ we would then know $u = \mu_0\tilde{\Lambda}u \in H_0^1(\mathcal{U})$.[2] But then we would have $u \in H_0^1(\mathcal{U})\backslash\{0\}$ which satisfies $u = \mu_0\Lambda u$, i.e.,

$$\tilde{B}\left(\frac{1}{\mu_0}u, v\right) = \langle u, v\rangle_{L^2} \qquad \text{for all } v \in H_0^1(\mathcal{U}),$$

i.e.,

$$\tilde{B}(u, v) - \mu_0\langle u, v\rangle_{L^2} = 0 \qquad \text{for all } v \in H_0^1(\mathcal{U}),$$

i.e.,

$$B(u, v) = 0 \qquad \text{for all } v \in H_0^1(\mathcal{U}).$$

By the weak maximum principle, however, and its uniqueness corollary, there is only one function $u \in H_0^1(\mathcal{U})$ for which

$$B(u, v) = 0 \qquad \text{for all } v \in H_0^1(\mathcal{U}).$$

That function is $u = 0 \in H_0^1(\mathcal{U})$. Thus, we have a contradiction, and alternative (ii) must be the one that holds.

Condition (ii) implies there is a unique weak solution to

$$\begin{cases} Lu = f & \text{on } \mathcal{U} \\ u\Big|_{\partial\mathcal{U}} = 0. \end{cases} \tag{1}$$

To see the existence, let $\tilde{f} = -\tilde{\Lambda}f/\mu_0$. This implies $u = \tilde{\Lambda}(\mu_0 u + f)$. In particular, since $\tilde{\Lambda} : L^2 \to H_0^1$, we now know $u \in H_0^1(\mathcal{U})$. Furthermore, rewriting what it means for $\tilde{\Lambda}(\mu_0 u + f) = u$, we have

$$\tilde{B}(u, v) = \langle \mu_0 u + f, v\rangle_{L^2} \qquad \text{for all } v \in H_0^1(\mathcal{U}).$$

That is,

$$\tilde{B}(u, v) - \mu_0\langle u, v\rangle_{L^2} = \langle f, v\rangle_{L^2} \qquad \text{for all } v \in H_0^1(\mathcal{U}),$$

[2]Note that this is a kind of regularity result; we start by knowing only that $u \in L^2$, but then use the fact that u satisfies some equation to show u has one weak derivative.

or

$$B(u,v) = \langle f, v\rangle_{L^2} \qquad \text{for all } v \in H_0^1(\mathcal{U}).$$

Thus, we have shown there exists some $u \in H_0^1(\mathcal{U})$ which is a weak solution of (1).

Uniqueness, in this case, follows from the corollary of the weak maximum principle below. □

1 Auxiliary results

Theorem 3 *(Fredholm's theorem) If $\Lambda : \mathcal{B} \to \mathcal{B}$ is a compact operator on a Banach space and $\lambda \neq 0$, then exactly one of the following holds*

(i) *λ is an eigenvalue for Λ, i.e., there is some $v \in \mathcal{B}\backslash\{0\}$ such that $\Lambda v = \lambda v$. In this case, $\Lambda - \Lambda I$ is neither one-to-one nor onto.*

(ii) *(λ is not an eigenvalue for Λ, and) for each $\xi \in \mathcal{B}$, there is some $x \in \mathcal{B}$ such that $\Lambda x - \lambda x = \xi$, i.e., $\Lambda - \lambda I$ is onto.*

In fact, in this case, $\Lambda - \lambda I$ is one-to-one and onto and $(\Lambda - \lambda I)^{-1}$ is bounded.

Theorem 4 *(weak maximum principle for weak subsolutions) Assume $c \geq 0$. If $u \in H^1(\Omega)$ and $Lu \leq 0$ in the sense that*

$$B(u,v) \leq 0 \qquad \text{for all } v \in C_c^\infty(\Omega) \text{ with } v \geq 0,$$

then

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+$$

where the supremum on the left is the essential supremum defined by

$$\inf\{M : \text{measure}\{x : u(x) \geq M\} = 0\},$$

$u^+ = \max\{u, 0\}$, *and the supremum on the right is taken in the trace sense:*

$$\inf\{M : (u^+ - M)^+ \in H_0^1(\Omega)\}.$$

One can also formulate a weak minimum principle for weak supersolutions and the two together have the following as a corollary:

Corollary 1 *(uniqueness of weak solutions) If L is uniformly elliptic and $c \geq 0$, then there is at most one weak solution $u \in H_0^1(\mathcal{U})$ of the equation $Lu = f$. In particular, if $u \in H_0^1(\mathcal{U})$ and $B(u,v) = 0$ for all $v \in H_0^1(\mathcal{U})$, then $u = 0$.*