Math 6342, Exam 1 Name and section: ____________________

1. (25 points) (Hamilton-Jacobi Equation) Solve the initial value problem

$$\begin{cases} u_t + u_x^2/2 = 0 \text{ on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = x^2. \end{cases}$$

Solution: The Hopf-Lax formula for this IVP is given by

$$u(x,t) = \min_{\xi \in \mathbb{R}^n} \left\{ tL\left(\frac{x-\xi}{t}\right) + \xi^2 \right\}$$

where $L = H^*$ is the convex dual of the Hamiltonian $H(p) = |p|^2/2$ appearing in the PDE. The formula for the convex dual in this case is given by

$$L(v) = \max_p \{pv - p^2/2\} = v^2/2.$$

Thus, the Hopf-Lax formula is

$$\begin{aligned} u(x,t) &= \min_\xi \left\{ \frac{t}{2}\left(\frac{x-\xi}{t}\right)^2 + \xi^2 \right\} \\ &= \min_\xi \left(1 + \frac{1}{2t}\right)\xi^2 - \frac{1}{t}x\xi + \frac{1}{2t}x^2. \end{aligned}$$

The minimum occurs when $(2 + 1/t)\xi - x/t = 0$, i.e., $\xi = x/(2t+1)$. That gives

$$u(x,t) = \frac{x^2}{2t+1}$$

which is the unique solution.

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2. (25 points) (separation of variables) Solve the boundary value problem

$$\begin{cases} \Delta u = 0 \quad \text{on } (-\pi,\pi)\times(0,2\pi) \\ u(x,0) = \sin x \\ u(\pm\pi, y) = 0 \\ u(x,2\pi) = \cos(x/2). \end{cases}$$

Solution: Setting $u = A(x)B(y)$, we find separated equations $A'' = -\lambda A$ and $B'' = \lambda B$ with boundary conditions $A(\pm\pi) = 0$. If $\lambda > 0$, then $A = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x$, and the boundary conditions give $a\cos\sqrt{\lambda}\pm b\sin\sqrt{\lambda} = 0$, or $a\cos\sqrt{\lambda} = b\sin\sqrt{\lambda} = 0$. If $a \neq 0$, then we get

$$\lambda = \lambda_j = \left(\frac{1}{2} + j\right)^2 \quad \text{for } j = 0, 1, \ldots$$

and $b = 0$. If $a = 0$, then we can assume $\sin\sqrt{\lambda} = 0$, so

$$\lambda = \tilde{\lambda}_j = j^2 \quad \text{for } j = 1, 2, \ldots$$

In view of the B equation, we get separated variables solutions

$$u_j(x,t) = a_j \sinh\left[\left(j + \frac{1}{2}\right)y - b_j\right]\cos\left(\frac{2j+1}{2}x\right) \quad \text{and} \quad \tilde{u}_j(x,t) = \tilde{a}_j \sinh(jy - \tilde{b}_j)\sin(jx).$$

As is often the case, we don't get any eigenfunctions in this situation for $\lambda \leq 0$, and we have enough already anyway.

The tricky part of this problem is to use superposition with the boundary values, taking first $\tilde{u}(x,0) = 0$ and then $u(x,2\pi) = 0$.

With the first modification of boundary values, we can use u_j with $j = 0$ to find

$$u_0 = \frac{\sinh\left(\frac{y}{2}\right)}{\sinh\pi}\cos\left(\frac{x}{2}\right).$$

The second modification leads to

$$\tilde{u}_1 = -\frac{\sinh(x - 2\pi)}{\sinh(2\pi)}\sin x.$$

The solution is then $u = u_0 + \tilde{u}_1$.

3. (25 points) (Fourier Transform) Consider the heat conduction model

$$\begin{cases} u_t = u_{xx} & \text{on } (0,\infty)\times(0,\infty) \\ u(x,0) = \begin{cases} 1-|x-1|, & 0 \le x \le 2 \\ 0, & x > 2, \end{cases} \\ u(0,t) = 0 & \text{for } t \ge 0. \end{cases}$$

Define the *Fourier sine transform* of a function f by

$$\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)\sin(\xi x)\,dx.$$

(i) Find an initial value problem satisfied by the spatial Fourier sine transform of a solution $u = u(x,t)$:

$$\tilde{u}(\xi,t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty u(x,t)\sin(\xi x)\,dx.$$

Hint: Assume $\lim_{x\to\infty} u_x(x,t) = \lim_{x\to\infty} u(x,t) = 0$.

(ii) Determine $\tilde{u}(\xi,t)$ by solving the initial value problem.

Solution:

(i)

$$\begin{aligned} \hat{u}_t &= \frac{1}{\sqrt{2\pi}} \int_0^\infty u_t \sin(\xi x)\,dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty u_{xx} \sin(\xi x)\,dx \\ &= -\frac{\xi}{\sqrt{2\pi}} \int_0^\infty u_x \cos(\xi x)\,dx \\ &= -\frac{\xi^2}{\sqrt{2\pi}} \int_0^\infty u \sin(\xi x)\,dx \\ &= -\xi^2 \tilde{u}. \end{aligned}$$

For the initial condition we find

$$\begin{aligned} \tilde{u}(\xi,0) &= \frac{1}{\sqrt{2\pi}} \left[\int_0^1 x\sin(\xi x)\,dx + \int_1^2 (2-x)\sin(\xi x)\,dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{\xi}\cos\xi + \frac{1}{\xi}\int_0^1 \cos(\xi x)\,dx + \frac{1}{\xi}\cos\xi - \frac{1}{\xi}\int_1^2 \cos(\xi x)\,dx \right] \\ &= \frac{1}{\xi^2\sqrt{2\pi}} \left[2\sin\xi - \sin(2\xi) \right]. \end{aligned}$$

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Thus, the initial value problem for $\tilde{u}$ (with parameter ξ) is:

$$\begin{cases} \frac{d\tilde{u}}{dt} = -\xi^2 \tilde{u} \quad \text{for } t > 0 \\ \tilde{u}(\xi, 0) = \frac{1}{\xi^2\sqrt{2\pi}} \left[2\sin\xi - \sin(2\xi)\right]. \end{cases}$$

(ii) The solution of the ODE is

$$\tilde{u}(\xi, t) = \frac{1}{\xi^2\sqrt{2\pi}} \left[2\sin\xi - \sin(2\xi)\right] e^{-\xi^2 t}.$$

The next step would be to use the inverse Fourier sine transform to determine an integral formula for the solution.

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4. (25 points) (product of Hölder continuous functions) Let Ω be a bounded domain with diameter d. Show that if $u \in C^{\alpha}(\bar{\Omega})$ and $v \in C^{\beta}(\bar{\Omega})$ for some $\alpha, \beta \in (0,1)$, then $uv \in C^{\gamma}(\bar{\Omega})$ with

$$|uv|_{C^{\gamma}} \leq C|u|_{C^{\alpha}}|v|_{C^{\beta}}$$

where $\gamma = \min\{\alpha, \beta\}$ and $C \geq 0$ is a constant. Bonus: Prove it with $C = \max\{1, d^{\alpha+\beta-2\gamma}\}$. Hint: Consider the cases $\alpha \leq \beta$ and $\beta \leq \alpha$.

Solution: Take the case, $\alpha \leq \beta$. Then $\gamma = \alpha$ and $\alpha + \beta - 2\gamma = \beta - \alpha$. Therefore, we are being asked to prove that for $\alpha \leq \beta$,

$$\begin{aligned} |uv|_{\infty} + [uv]_{\alpha} &\leq C\left(|u|_{\infty} + [u]_{\alpha}\right)\left(|v|_{\infty} + [v]_{\beta}\right) \\ &= C\left(|u|_{\infty}|v|_{\infty} + |u|_{\infty}[v]_{\beta} + [u]_{\alpha}|v|_{\infty} + [u]_{\alpha}[v]_{\beta}\right). \end{aligned}$$

Since the supremum of a product is less than or equal to the product of the suprema of the factors,

$$|uv|_{\infty} \leq |u|_{\infty}|v|_{\infty}$$

Also, the sup of a sum is less than or equal to the sum of the sups, so

$$\begin{aligned} [uv]_{\alpha} &= \sup \frac{|u(x)v(x) - u(\xi)v(\xi)|}{|x-\xi|^{\alpha}} \\ &= \sup \frac{|u(x)v(x) - u(x)v(\xi) + u(x)v(\xi) - u(\xi)v(\xi)|}{|x-\xi|^{\alpha}} \\ &\leq \sup \left(|u(x)| \frac{|v(x) - v(\xi)|}{|x-\xi|^{\beta}} \; |x-\xi|^{\beta-\alpha} + \frac{|u(x) - u(\xi)|}{|x-\xi|^{\alpha}} |v(\xi)| \right) \\ &\leq \sup \left(|u(x)| \frac{|v(x) - v(\xi)|}{|x-\xi|^{\beta}} \; d^{\beta-\alpha} + \frac{|u(x) - u(\xi)|}{|x-\xi|^{\alpha}} |v(\xi)| \right) \\ &\leq \sup \left(|u(x)| \frac{|v(x) - v(\xi)|}{|x-\xi|^{\beta}} \; d^{\beta-\alpha} \right) + \sup \left(\frac{|u(x) - u(\xi)|}{|x-\xi|^{\alpha}} |v(\xi)| \right) \\ &\leq d^{\beta-\alpha}|u|_{\infty}[v]_{\beta} + [u]_{\alpha}|v|_{\infty}. \end{aligned}$$

Thus, just by adding the final product, $[u]_{\alpha}[v]_{\beta}$, we have

$$|uv|_{\infty} + [uv]_{\alpha} \leq |u|_{\infty}|v|_{\infty} + d^{\beta-\alpha}|u|_{\infty}[v]_{\beta} + [u]_{\alpha}|v|_{\infty} + [u]_{\alpha}[v]_{\beta}.$$

Now we recall that $d^{\beta-\alpha} = d^{\alpha+\beta-2\gamma}$ in this case. Therefore, we can replace the coefficient of each of the four terms with C:

$$|uv|_{\infty} + [uv]_{\alpha} \leq C|u|_{\infty}|v|_{\infty} + C|u|_{\infty}[v]_{\beta} + C[u]_{\alpha}|v|_{\infty} + C[u]_{\alpha}[v]_{\beta}.$$

This is what we needed to prove, and since the other case is symmetric, we are done.