Math 6342, Exam 1 (practice) Name and section: ____________________

1. (25 points) (Hamilton-Jacobi Equation) Solve the initial value problem

$$\begin{cases} u_t + u_x^2/2 = 0 \text{ on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = x^2. \end{cases}$$

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2. (25 points) (separation of variables) Solve the boundary value problem

$$\begin{cases} \Delta u = 0 \quad \text{on } (-\pi, \pi) \times (0, 2\pi) \\ u(x, 0) = \sin x \\ u(\pm\pi, y) = 0 \\ u(x, 2\pi) = \cos(x/2). \end{cases}$$

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3. (25 points) (Fourier Transform) Consider the heat conduction model

$$\begin{cases} u_t = u_{xx} & \text{on } (0,\infty)\times(0,\infty) \\ u(x,0) = \begin{cases} |x-1|, & 0 \le x \le 2 \\ 0, & x > 2, \end{cases} \\ u(0,t) = 0 & \text{for } t \ge 0. \end{cases}$$

Define the *Fourier sine transform* of a function f by

$$\tilde{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)\sin(\xi x)\,dx.$$

(i) Find an initial value problem satisfied by the spatial Fourier transform of a solution $u = u(x,t)$:

$$\tilde{u}(\xi,t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty u(x,t)\sin(\xi x)\,dx.$$

Hint: Assume $\lim_{x\to\infty} u_x(x,t) = \lim_{x\to\infty} u(x,t) = 0$.

(ii) Determine $\tilde{u}(\xi,t)$ by solving the initial value problem.

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4. (25 points) (product of Hölder continuous functions) Let Ω be a bounded domain with diameter d. Show that if $u \in C^{\alpha}(\bar{\Omega})$ and $v \in C^{\beta}(\bar{\Omega})$ for some $\alpha, \beta \in (0,1)$, then $uv \in C^{\gamma}(\bar{\Omega})$ with

$$|uv|_{C^{\gamma}} \leq C|u|_{C^{\alpha}}|v|_{C^{\beta}}$$

where $\gamma = \min\{\alpha, \beta\}$ and $C \geq 0$ is a constant. Bonus: Prove it with $C = \max\{1, d^{\alpha+\beta-2\gamma}\}$. Hint: Consider the cases $\alpha \leq \beta$ and $\beta \leq \alpha$.