Coercivity and the Poincaré inequality

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Coercivity for the bilinear form

$$B(u,v) = \int_\Omega \sum_{i,j} a_{ij} D_j u D_i v + \int_\Omega \sum_j b_j v D_j u + \int_\Omega cuv.$$

associated with the linear partial differential operator

$$Lu = -\sum_{i,j} D_i(a_{ij} Dju) + \sum_j b_j D_j u + cu$$

is the requirement that for some $m > 0$,

$$B(u,u) \geq m \|u\|_{H^1}^2.$$

Here we prove carefully the main lemma concerning coercivity for operators of the form L which are *elliptic* and explain the role played by the Poincaré inequality.

1 Ellipticity

We assume the coefficients a_{ij}, b_j and c all defined and bounded on the closure of some bounded domain $\Omega \subset \mathbb{R}^n$. We assume further the condition of *uniform ellipticity*, namely that for some $\epsilon_0 > 0$

$$\sum a_{ij} \xi_i \xi_j \geq \epsilon_0 |\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n.$$

Using ellipticity, we get the initial estimate

$$B(u,u) \geq \epsilon_0 \int |Du|^2 - \bar{b} \sum \int |u||D_j u| - \bar{c} \int |u|^2$$

where

$$\bar{b} = \sup_{j,x\in\Omega} |b_j(x)| \qquad \text{and} \qquad \bar{c} = \sup_{x\in\Omega} |c(x)|.$$

The last two terms are not in our favor. We only have the (small) $\epsilon_0\|Du\|_{L^2}^2$ term on which to rely. To make matters worse, we need to somehow insert an additive term $\|u\|_{L^2}^2$ on the right to get, finally, and H^1 norm on the right.

Let us first note that the inequality

$$ab \leq \frac{\epsilon^2}{2}a^2 + \frac{1}{2\epsilon^2}b^2$$

can be applied to the second term to preserve at least some of our only help. That is, for any $\epsilon > 0$,

$$B(u,u) \geq \epsilon_0 \int |Du|^2 - \frac{\bar{b}\epsilon^2}{2}\int |Du|^2 - \frac{\bar{b}}{2\epsilon^2}\int |u|^2 - \bar{c}\int |u|^2.$$

In particular, taking $\epsilon^2 < \epsilon_0/\bar{b}$, we get an inequality

$$B(u,u) \geq \frac{\epsilon_0}{2}\int |Du|^2 - M\int |u|^2 \tag{1}$$

where $M > 0$ is some (large) constant. Of course if there were no b and c terms there would be no troublesome $M\|u\|_{L^2}^2$ term, but we would still have the difficulty of replacing the norm of Du with an H^1 norm of u. We attempt to address this unavoidable difficulty now.

2 Poincaré inequality

Recall that the H^1 norm may be defined variously by

$$\|u\|_{H^1} = |u|_{L^2} + \sum_j |D_j u|_{L^2}$$

or

$$\|u\|_{H^1} = \left(|u|_{L^2}^2 + \sum_j |D_j u|_{L^2}^2\right)^{1/2}$$

or even

$$\|u\|_{H^1} = |u|_{L^2} + \max_j |D_j u|_{L^2}.$$

In view of our initial estimate above, it looks like we might wish to use the second form of the norm.

There are various inequalities which relate/bound norms of a function in terms of norms of its derivative. Perhaps the simplest is the C_c^∞ *Sobolev inequality*:

> *If $u \in C_c^\infty(\mathbb{R}^n)$ and $1 \le p < n$, then*
>
> $$\|u\|_{L^{p^*}} \le C\|Du\|_{L^p}$$
>
> *where $p^* = np/(n-p)$ is the* Sobolev exponent. *Here C is a positive constant that depends on n and p, but (most importantly) is independent of u.*

In fact, one only needs $u \in C_c^1(\mathbb{R}^n)$ for this result.

In this case, we have $u \in H_0^1(\Omega)$, so we use the following version called the $W_0^{1,p}$ Poincaré inequality:

Theorem 1 *If Ω is a bounded domain in $\mathbb{R}^n$, $n > p \ge 1$, and $1 \le q \le p^*$, then there is a constant $C = C(n, p, q, \Omega)$ such that*

$$\|u\|_{L^q(\Omega)} \le C\|Du\|_{L^p(\Omega)} \qquad \text{for all } u \in W_0^{1,p}(\Omega).$$

Proof: Since C_c^∞ is dense in $W_0^{1,p}$, there is a sequence of C_c^∞ functions u_j with

$$\|u_j - u\|_{W^{1,p}} \to 0.$$

Setting

$$\bar{u}_j(x) = \begin{cases} u_j(x), & x \in \Omega \\ 0, & x\mathbb{R}^n \backslash \Omega \end{cases}$$

we have $\bar{u}_j \in C_c^\infty(\mathbb{R}^n)$. Therefore, applying the C_c^∞ Sobolev inequality, we get

$$\|\bar{u}_j\|_{L^{p^*}} \le C\|D\bar{u}_j\|_{L^p}.$$

This is precisely the same as

$$\|u_j\|_{L^{p^*}(\Omega)} \le C\|Du_j\|_{L^p(\Omega)}.$$

And we can take a limit to obtain

$$\|u\|_{L^{p^*}(\Omega)} \le C\|Du\|_{L^p(\Omega)}.$$

Finally, we claim that for $1 \le q \le p^*$ there is some C for which

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{L^{p^*}(\Omega)}.$$

To see this, note that since $|u|^q \in L^m$ where $m = p^*/q \ge 1$,

$$\begin{aligned}
\|u\|^q_{L^q(\Omega)} &= \int \left(|u|^{p^*}\right)^{q/p^*} \\
&= \int \left(|u|^{p^*}\right)^{q/p^*} \chi_\Omega \\
&\le \left(\int |u|^{p^*}\right)^{q/p^*} |\Omega|^{\frac{m}{m-1}} \\
&= \left(\int |u|^{p^*}\right)^{q/p^*} |\Omega|^{\frac{p^*}{p^*-q}}.
\end{aligned}$$

Thus,

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{L^{p^*}(\Omega)} \quad \text{with } C = |\Omega|^{\frac{p^*}{q(P^*-q)}}.$$

3 Estimate

Returning to (1) and using Theorem 1 in the form $\|Du\|_{L^p(\Omega)} \ge \|u\|_{L^q(\Omega)}/C$, we obtain

$$\begin{aligned}
B(u,u) &\ge \frac{\epsilon_0}{4}\int |Du|^2 + \frac{\epsilon_0}{4}\int |Du|^2 - M\int |u|^2 \\
&\ge \frac{\epsilon_0}{4}\int |Du|^2 + \frac{\epsilon_0}{4C}\int |u|^2 - M\int |u|^2 \\
&\ge m\|u\|^2_{H^1(\Omega)} - M\int |u|^2
\end{aligned}$$

where

$$m = \min\left\{\frac{\epsilon_0}{4}, \frac{\epsilon_0}{4C}\right\} > 0.$$

That is essentially the best we can do:

Lemma 1 *(Main coercivity lemma) If L is elliptic, then there is some constants $m, M > 0$ such that*

$$B(u,u) \ge m\|u\|^2_{H^1(\Omega)} - M\int |u|^2.$$

Corollary 1 *If L is elliptic, then there is a constant $M > 0$ such that $\tilde{B}(u,v) = B(u,v) + \mu\langle u,v\rangle_{L^2}$ is coercive for each $\mu \geq M$.*

Corollary 2 *If L is elliptic, and*

$$\int_\Omega \sum_j b_j u D_j u + \int_\Omega c u^2 \geq 0 \qquad \text{for } u \in H_0^1(\Omega),$$

then $B : H_0^1 \times H_0^1 \to \mathbb{R}$ is coercive.