Bounded Linear Operators

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Here is a bit more careful treatment of the inequalities/estimates leading to the conclusion that a second order linear partial differential operator in divergence form has associated with it a bounded bilinear form.

1 The players

$$Lu = -\sum_{i,j} D_i(a_{ij}Dju) + \sum_j b_j D_j u + cu$$

with a_{ij}, b_j and c all defined and bounded on the closure of some domain Ω. The customary (Evans') minus sign makes no difference in this discussion, nor do we need to assume ellipticity.

Assuming $u, v \in C^2(\bar{\Omega})$ and v has zero boundary values, we can multiply by v and integrate to get the bilinear form

$$B(u,v) = \int_\Omega \sum_{i,j} a_{ij} D_i u D_j v + \int_\Omega \sum_j b_j v D_j u + \int_\Omega uv.$$

(Yes, I've switched the indices in the first term, but we'll assume the top order coefficients a_{ij} are symmetric, so that's OK.)

So these are the main players. It is obvious that B extends to $H_0^1(\Omega) \times H_0^1(\Omega)$, and we want to show it is bounded, i.e.,

$$|B(u,v)| \leq C\|u\|_{H^1}\|v\|_{H^1}.$$

2 The norms

The H^1 norm is defined variously by

$$\|u\|_{H^1} = |u|_{L^2} + \sum_j |D_j u|_{L^2}$$

or

$$\|u\|_{H^1} = \left(|u|_{L^2}^2 + \sum_j |D_j u|_{L^2}^2 \right)^{1/2}$$

or even

$$\|u\|_{H^1} = |u|_{L^2} + \max_j |D_j u|_{L^2}.$$

Let's briefly check that these norms are *equivalent.* This means, for example in the case of the first two norms, that there are positive constants c and C such that

$$c\left(|u|_{L^2} + \sum_j |D_j u|_{L^2} \right) \leq \left(|u|_{L^2}^2 + \sum_j |D_j u|_{L^2}^2 \right)^{1/2} \leq C\left(|u|_{L^2} + \sum_j |D_j u|_{L^2} \right).$$

Notice first that the relation of being equivalent norms is symmetric. In fact, taking $\tilde{c} = 1/C$ and $\tilde{C} = 1/c$, we have

$$\tilde{c}\left(|u|_{L^2}^2 + \sum_j |D_j u|_{L^2}^2 \right)^{1/2} \leq |u|_{L^2} + \sum_j |D_j u|_{L^2} \leq \tilde{C}\left(|u|_{L^2}^2 + \sum_j |D_j u|_{L^2}^2 \right)^{1/2}.$$

As to the actual equivalence of these norms,

$$\begin{aligned}
\left(|u|_{L^2} + \sum_j |D_j u|_{L^2} \right)^2 &= |u|_{L^2}^2 + 2\sum_j |u|_{L^2}|D_j u|_{L^2} + \left(\sum |D_j u|_{L^2} \right)^2 \\
&= |u|_{L^2}^2 + 2\sum_j |u|_{L^2}|D_j u|_{L^2} + \sum_{i \neq j} |D_i u|_{L^2}|D_j u|_{L^2} + \sum |D_j u|_{L^2}^2.
\end{aligned}$$

Applying the fact that $2ab \leq a^2 + b^2$ to the first summation term, we see

$$2\sum_j |u|_{L^2}|D_j u|_{L^2} \leq \sum_j \left(|u|_{L^2}^2 + |D_j u|_{L^2}^2 \right) \leq n|u|_{L^2}^2 + \sum |D_j u|_{L^2}^2.$$

(If I've counted correctly) the same thing applied to the next summation term gives

$$\sum_{i\neq j} |D_i u|_{L^2} |D_j u|_{L^2} \leq \frac{1}{2} \sum_{i\neq j} \left(|D_i u|_{L^2}^2 + |D_j u|_{L^2}^2 \right) \leq 2n \sum |D_j u|_{L^2}^2.$$

Thus returning to our original string of inequalities

$$\begin{aligned}\left(|u|_{L^2} + \sum_j |D_j u|_{L^2} \right)^2 &\leq (n+1)|u|_{L^2}^2 + (2n+2) \sum |D_j u|_{L^2}^2 \\ &\leq 2(n+1) \left(|u|_{L^2}^2 + \sum |D_j u|_{L^2}^2 \right).\end{aligned}$$

Thus, the first inequality required holds with $\tilde{C} = \sqrt{2(n+1)}$. For the second one,

$$|u|_{L^2}^2 + \sum |D_j u|_{L^2}^2 \leq \left(|u|_{L^2} + \sum_j |D_j u|_{L^2} \right)^2$$

simply because all the cross terms in the expansion of the right side are nonnegative. Thus, we can take $C = 1$. We have shown that the first two norms are equivalent.

The last norm is clearly always smaller than (or equal to) the first one. Thus, we note that

$$|u|_{L^2} + \sum_j |D_j u|_{L^2} \leq |u|_{L^2}^2 + n \max_j |D_j u|_{L^2}^2 \leq n \left(|u|_{L^2}^2 + \max_j |D_j u|_{L^2}^2 \right),$$

and we have shown the first norm is equivalent to the last. I'll leave it as an exercise to show that the relation of equivalence of norms is transitive, so the last possible equivalence of these three norms follows as well.

3 Estimates

If we take the absolute value of $B(u, v)$ and apply the triangle inequality, we get three terms to estimate. Moving the absolute values inside the integrals and applying the triangle inequality some more, these terms are

$$|B(u, v)| \leq \int \sum |a_{ij}||D_i u||D_j v| + \int \sum |b_j||v||D_j u| + \int |c||u||v|.$$

Let's take the last term first. The Cauchy-Schwarz inequality on L^2 gives

$$\int |c||u||v| \leq C \int |u||v| = C\langle |u|, |v|\rangle_{L^2} \leq C|u|_{L^2}|v|_{L^2}.$$

where

$$C = \sup_{x \in \Omega} |c(x)|.$$

We could go ahead and use $ab \leq (a^2 + b^2)/2$, but we'll leave that as it is for now.

The second term satisfies

$$\int |b_j||v||D_j u| \leq B \sum \int |v||D_j u| \leq B \sum |v|_{L^2}|D_j u|_{L^2} = B|v|_{L^2} \sum |D_j u|_{L^2}$$

where

$$B = \sup_{j, x \in \Omega} |b_j(x)|.$$

Since these last two factors are both terms in the first norm above (one for v and one for u), we'll leave these as they are.

Finally, the first term can be estimated like this:

$$\sum \int |a_{ij}||D_i u||D_j v| \leq A \sum |D_i u|_{L^2}|D_j v|_{L^2} \leq A \left(\sum_i |D_i u|_{L^2}\right)\left(\sum_j |D_j v|_{L^2}\right)$$

where

$$A = \sup_{i,j,x \in \Omega} |a_{ij}|.$$

Letting $M = \max\{A, B, C\}$, we have an estimate

$$\begin{aligned}
|B(u,v)| &\leq M \left[\left(\sum_i |D_i u|_{L^2}\right)\left(\sum_j |D_j v|_{L^2}\right) + |v|_{L^2} \sum |D_j u|_{L^2} + |u|_{L^2}|v|_{L^2}\right] \\
&\leq M \left[\left(\sum_i |D_i u|_{L^2}\right)\left(\sum_j |D_j v|_{L^2}\right) + |v|_{L^2} \sum |D_j u|_{L^2}\right. \\
&\qquad \left. + |u|_{L^2} \sum |D_j v|_{L^2} + |u|_{L^2}|v|_{L^2}\right] \\
&= M \left(\sum_i |D_i u|_{L^2} + |u|_{L^2}\right)\left(\sum_j |D_j v|_{L^2} + |v|_{L^2}\right) \\
&= M \left(|u|_{H^1} + |v|_{H^1}\right)
\end{aligned}$$

in terms of the first H^1 norm given above. Thus, B is bounded.

One can try to give a "tighter" estimate in terms of different integral bounds for the a_{ij} as I did in class, and you might want/need to do this if you want to use slightly more general coefficients. For example,

$$\sum_i \int |D_i u| \sum_j |a_{ij}||D_j v| \le \sum_i \int |D_i u| \left(\sum_j a_{ij}^2 \right)^{1/2} |Dv|$$

where we have applied the Schwarz inequality in $\mathbb{R}^n$ to the inner summation considered as a simple dot product. From there, if we have an L^∞ bound on $\sum_{ij} a_{ij}^2$, we can get

$$\sum_i \int |D_i u| \sum_j |a_{ij}||D_j v| \le A \sum_i |D_i u|_{L^2} |Dv|_{L^2} = A|Dv|_{L^2} \sum_i |D_i u|_{L^2}$$

which I claimed was bounded by a constant times

$$\left(\sum_i |D_i u|_{L^2} \right) \left(\sum_j |D_j v|_{L^2} \right).$$

In fact,

$$|Dv|_{L^2}^2 = \int \sum |D_j v|^2 \le \int \left(\sum |D_j v| \right)^2.$$

(This is because of the cross terms again.) Thus, taking a square root

$$|Dv|_{L^2} \le \left| \sum |D_j v| \right|_{L^2}.$$

And we get what was claimed with the same constant by the triangle inequality.