**Math 6341, Final Exam: Various Topics (practice)** __________________

1. (25 points) (3.5.9-10, Hamilton-Jacobi PDE)

   **(a)** Find the convex dual (Legendre transform) $L = H^*$ for the Hamiltonian

   $$H(p) = |p|^3.$$

   **(b)** Write down the Hopf-Lax formula and explain its relation to the variational problem of minimizing

   $$\int_0^t L(w'(\sigma))\, d\sigma + u_0(w(0))$$

   for a given Lagrangian $L$ and initial function $u_0$.

   **(c)** Find a weak solution of the Hamilton-Jacobi IVP

   $$\begin{cases} u_t + |u_x|^3 = 0, & (x,t) \in \mathbb{R} \times (0, \infty) \\ u(x,0) = |x|. \end{cases}$$

   **(d)** Is the weak solution you found unique? (Explain)

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**Solution:**

**(a)** Recall that

$$H^*(v) = \max_p \{p \cdot v - |p|^3\}.$$

Taking the gradient with respect to $p$ to find a critical point, we get

$$v - 3|p|p = 0.$$

It follows that $|p| = \sqrt{|v|/3}$ and therefore the critical point is $p = v/\sqrt{3|v|}$. The resulting maximum value is

$$L(v) = |v|^{3/2}/\sqrt{3} - (|v|/3)^{3/2} = c|v|^{3/2}$$

where $c = 2/(3\sqrt{3})$.

As a partial aside, the Legendre transform is well-defined if $H(p) = |p|^3$ is convex and

$$\lim_{|p|\to\infty} \frac{|p|^3}{|p|} = +\infty.$$

The second condition clearly holds. One way to check convexity, is to compute the Hessian $D^2H$, and consider the quadratic form $\langle D^2H\xi, \xi\rangle$. In fact, $H_{p_i} = 3p_i|p|$, and

$$H_{p_i p_j} = 3\left(\delta_{ij}|p| + \frac{p_i p_j}{|p|}\right).$$

Thus,

$$\sum_{ij} H_{p_i p_j}\xi_i\xi_j = 3\left(|p||\xi|^2 + \frac{1}{|p|}\sum p_i p_j \xi_i \xi_j\right) \geq 3(|p| - |p|)|\xi|^2 > 0.$$

This means $D^2H$ is positive semidefinite, which means $H$ is convex. Taking $p_1 = -p_2$ and all the other $p_j = 0$ and $\xi_1 = \xi_2$ with all the other $\xi_j = 0$, we find

$$\sum_{ij} H_{p_i p_j}\xi_i\xi_j = 3\left(|p||\xi|^2 - \frac{1}{|p|}\frac{|p|^2|\xi|^2}{2}\right) = \frac{3|p||\xi|^2}{2}.$$

This indicates that $H$ is not uniformly convex around $p = 0$.

Notice also that the expressions for $H_{p_i}$ and $H_{p_i p_j}$ are continuous, so $H \in C^2$. I believe this is enough regularity (and $H$ is even more regular) to make the arguments in our introduction to Hamilton-Jacobi equations, though in the discussion of the Hopf-Lax formula and weak solutions (p. 123) we formally assumed $H$ was "smooth" which presumably means $C^\infty$.

**(b)** If $L, u_0 : \mathbb{R}^n \to \mathbb{R}$ are given with $L$ convex and satisfying

$$\lim_{|v|\to\infty} \frac{L(v)}{|v|} = +\infty$$

and $u_0$ Lipschitz continuous, then

$$u(x,t) = \min_{\xi\in\mathbb{R}^n}\left\{tL\left(\frac{x-\xi}{t}\right) + u_0(\xi)\right\}$$

is the value of

$$\inf_{w\in C^1[0,t]}\left\{\int_0^t L(w'(\sigma))\,d\sigma + u_0(w(0)) : w(t) = x\right\}.$$

**(c)** According to part (a), the solution should be given by the Hopf-Lax formula

$$\begin{aligned} u(x,t) &= \min_{\xi\in\mathbb{R}^n} \frac{2t}{3\sqrt{3}}\left(\frac{|x-\xi|}{t}\right)^{3/2} + |\xi| \\ &= \min_{\xi\in\mathbb{R}^n} \frac{2}{3\sqrt{3t}}|x-\xi|^{3/2} + |\xi|. \end{aligned}$$

Taking the gradient (derivative since $n = 1$) with respect to $\xi$ to find critical points, we have

$$-\sqrt{\frac{|x-\xi|}{3t}}\frac{x-\xi}{|x-\xi|} + \frac{\xi}{|\xi|} = 0.$$

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It follows that any critical $\xi$ satisfies

$$|x - \xi| = 3t.$$

In particular, any associated minimum value will satisfy

$$u(x,t) = 2t + |\xi|. \tag{1}$$

Also in this case, we see that $\xi = x \pm 3t$. Thus, we find an initial candidate

$$u_1(x,t) = \begin{cases} 2t + |x - 3t|, & x \geq 0 \\ 2t + |x + 3t|, & x \leq 0. \end{cases}$$

There is another possibility, however. Namely, if we look at the gradient calculation above, there may be singular minima associated with $|\xi| = 0$ or $|x - \xi| = 0$. These possibilities give

$$u_2(x,t) = \frac{2}{3\sqrt{3t}}|x|^{3/2} \quad \text{and} \quad u_3(x,t) = |x|.$$

Thus, the Hopf-Lax formula reduces to

$$u(x,t) = \min_{t>0}\{u_1(x,t), u_t(x,t), u_3(x,t)\}.$$

If we consider first the region where $x > 3t$, we find

$$u_1 = x - t < u_3 = x \quad \text{and} \quad u_2 = \frac{2}{3\sqrt{3t}}x^{3/2}.$$

Thus, $u_3$ is ruled out in this region (which we should expect since $u_3$ doesn't satisfy the equation (n.b., Theorem 5), and near $t = 0$ we see that $u_1$ definitely provides the minimum. Furthermore,

$$\frac{\partial}{\partial t}[u_1 - u_2] = -1 + \frac{1}{3\sqrt{3}}\left(\frac{x}{t}\right)^{3/2}.$$

It will be found that this expression has a unique zero at $x = 3t$ where $u_1 = 2t = u_2$. Thus, $u_1$ is the minimum in the region $x > 3t$, and $u$ takes the value $2t$ along the line $x = 3t$.

Moving into the region $x < 3t$, we see that $u_1 = 5t - x$ ceases to be a solution of the PDE, and not surprisingly, we can see that it does not provide the minimum value. In fact, we still have $u_1 = u_2$ along $x = 3t$, and

$$\frac{\partial}{\partial t}[u_1 - u_2] = 5 + \frac{1}{3\sqrt{3}}\left(\frac{x}{t}\right)^{3/2} > 0.$$

Thus, we need only make the comparison between $u_2$ (the presumed solution) and $u_3 = x$. As noted above, $x = u_3 > u_1 = u_2$ along $x = 3t$. Furthermore, setting $u_2 = u_3$ yields $x = (27/4)t$ which is actually in the region $x > 3t$. We conclude that $u_2 < u_3$ in the region $x < 3t$ and (with a similar analysis for $x \le 0$)

$$u(x,t) = \begin{cases} -t + |x|, & |x| \ge 3t \\ \frac{2}{3\sqrt{3t}}|x|^{3/2}, & |x| < 3t. \end{cases}$$

On the smooth regions, this solves the PDE in accordance with Theorem 5, though at least formally, we assumed $H$ was smooth in the proof of that theorem, and the $H$ considered in this problem has an apparent singularity at the origin—though I guess it is at least $C^3$.

**(d)** Uniqueness of the weak solution follows from uniform convexity of $H$ or semi-concavity of $u_0(x) = |x|$. (Theorem 8). Since $|x|$ is not semi-concave, we compute

$$H_{pp} = (3p|p|)' = 3(|p| + p^2/|p|) = 6|p|.$$

This value is not bounded away from zero, so $H$ is not uniformly convex. We cannot conclude uniqueness of the solution above.

2. (25 points) (Green's Functions for Laplace's PDE, §2.2.4)

**(a)** Write down the fundamental solution $\Phi = \Phi(x)$ for Laplace's PDE (centered at $x = 0$).

**(b)** Compute $D\Phi(x)$.

**(c)** Compute

$$\lim_{r\to 0} \int_{\{\xi : |\xi - x| = r\}} u(\xi) e^{-ix\cdot(\xi - x)} D\Phi(\xi - x) \cdot n$$

where $n$ is the outward normal to $B_r(x)$.

**Solution:**

**(a)**

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & n = 2 \\ \frac{1}{n(n-2)\omega_n}\frac{1}{|x|^{n-2}}, & n > 2. \end{cases}$$

**(b)**

$$D\Phi(x) = -\frac{1}{n\omega_n}\frac{x}{|x|^n}.$$

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**(c)**

$$
\begin{aligned}
&\lim_{r\to 0}\int_{\{\xi:|\xi-x|=r\}} u(\xi)e^{-ix\cdot(\xi-x)}D\Phi(\xi-x)\cdot n\\
&= \lim_{r\to 0} -\frac{1}{n\omega_n}\int_{|\xi-x|=r} u(\xi)e^{-ix\cdot(\xi-x)}\frac{\xi-x}{|\xi-x|^n}\cdot\frac{\xi-x}{|\xi-x|}\\
&= -\frac{1}{n\omega_n r^{n-1}}\lim_{r\to 0}\int_{|\xi-x|=r} u(\xi)e^{-ix\cdot(\xi-x)}\\
&= -\frac{1}{|\partial B_r|}\lim_{r\to 0}\int_{|\xi-x|=r} u(\xi)e^{-ix\cdot(\xi-x)}\\
&= -u(\xi)e^{-ix\cdot(\xi-x)}\Big|_{\xi=x}\\
&= -u(x).
\end{aligned}
$$

3. (25 points) (4.7.2) Find a separated variables solution of

$$
\begin{cases} \Delta u = 0 \text{ on } \mathbb{R}^2 \\ u(x,0)=0, \quad u_y(x,0)=\sin x. \end{cases}
$$

Explain your reasoning carefully.

**Solution:** Setting $u = f(x)g(y)$, we obtain away from $f=0$ or $g=0$ a separation constant $\lambda$ such that

$$-\frac{f''}{f} = \frac{g''}{g} = \lambda.$$

Thus, we obtain two ODEs

$$f'' = -\lambda f \quad \text{and} \quad g'' = \lambda g.$$

The first boundary condition gives $f(x)g(0)=0$ from which we conclude $g(0)=0$, since $f(x)=0$ cannot lead to a solution satisfying the other boundary condition.

The second boundary condition is $f(x)g'(0)=\sin x$. Therefore,

$$f(x) = \frac{\sin x}{g'(0)}.$$

It follows from this that $f''=-f$. In view of the first ODE, we must have $\lambda = 1$. The second ODE with the first boundary condition then yeilds

$$g(y) = g'(0)\sinh y.$$

The solution is thus,

$$u(x,y) = f(x)g(y) = \sinh y \sin x.$$

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4. (25 points) (Fourier Transform, §4.3.1)

**(a)** Define the Fourier transform of $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

**(b)** If $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, show

$$(u * v)^\wedge = (2\pi)^{n/2} \hat{u}\hat{v}.$$

**Solution:**

**(a)**

$$\hat{u} = \frac{1}{(2\pi)^{n/2}} \int_{x \in \mathbb{R}^n} e^{-ix\cdot\xi} u(x).$$

**(b)**

$$\begin{aligned}
(u * v)^\wedge(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{x \in \mathbb{R}^n} e^{-ix\cdot\xi} (u * v)(x) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{x \in \mathbb{R}^n} e^{-ix\cdot\xi} \left( \int_{\eta \in \mathbb{R}^n} u(x-\eta) v(\eta) \right) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\eta \in \mathbb{R}^n} \left( \int_{x \in \mathbb{R}^n} e^{-ix\cdot\xi} u(x-\eta) v(\eta) \right) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\eta \in \mathbb{R}^n} \left( \int_{x \in \mathbb{R}^n} e^{-ix\cdot\xi} u(x-\eta) \right) v(\eta) \\
&= \int_{\eta \in \mathbb{R}^n} \left( \frac{1}{(2\pi)^{n/2}} \int_{\tilde{x} \in \mathbb{R}^n} e^{-i(\tilde{x}+\eta)\cdot\xi} u(\tilde{x}) \right) v(\eta) \\
&= \int_{\eta \in \mathbb{R}^n} e^{-i\eta\cdot\xi} \left( \frac{1}{(2\pi)^{n/2}} \int_{\tilde{x} \in \mathbb{R}^n} e^{-i\tilde{x}\cdot\xi} u(\tilde{x}) \right) v(\eta) \\
&= \int_{\eta \in \mathbb{R}^n} e^{-i\eta\cdot\xi} \hat{u}(\xi) v(\eta) \\
&= \hat{u}(\xi) \int_{\eta \in \mathbb{R}^n} e^{-i\eta\cdot\xi} v(\eta) \\
&= (2\pi)^{n/2} \hat{u}(\xi)\hat{v}(\xi).
\end{aligned}$$