Math 6341, Exam 2: 2.3-2.4 (practice) Name and section: ____________________

1. (25 points) Solve the initial/boundary value problem

$$\begin{cases} u_t = \Delta u \text{ on } [-1,1]\times(0,\infty) \\ u(-1,t) = u(1,t) = 0 \\ u(x,0) = \sin(\pi x). \end{cases}$$

Hint: Solve on $\mathbb{R}\times(0,\infty)$ using spatial convolution. Then show you have also solved this problem.

Solution: A spatial convolution solution is

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi t}}\int_{\xi\in\mathbb{R}} e^{-\frac{|x-\xi|^2}{4t}}\sin(\pi\xi) \\ &= \frac{1}{\sqrt{4\pi t}}\int_{\eta\in\mathbb{R}} e^{-\frac{|\eta|^2}{4t}}\sin(\pi(x-\eta)). \end{aligned}$$

According to Theorem 1 on page 47, this is a solution of the PDE on $\mathbb{R}\times[0,\infty)$ satisfying the initial condition.

From the second integral expression, we have

$$\begin{aligned} u(-1,t) &= \frac{1}{\sqrt{4\pi t}}\int_{\eta\in\mathbb{R}} e^{-\frac{|\eta|^2}{4t}}\sin(\pi(-1-\eta)) \\ &= \frac{1}{\sqrt{4\pi t}}\int_{\eta\in\mathbb{R}} e^{-\frac{|\eta|^2}{4t}}\sin(\eta)) \\ &= \frac{1}{\sqrt{4\pi t}}\left[\int_{-\infty}^{0} e^{-\frac{|\eta|^2}{4t}}\sin(\eta))\,d\eta + \int_0^\infty e^{-\frac{|\eta|^2}{4t}}\sin(\eta))\,d\eta\right] \\ &= \frac{1}{\sqrt{4\pi t}}\left[\int_{\infty}^{0} e^{-\frac{|\xi|^2}{4t}}\sin(-\xi))\,(-1)d\xi + \int_0^\infty e^{-\frac{|\eta|^2}{4t}}\sin(\eta))\,d\eta\right] \\ &= 0. \end{aligned}$$

The other boundary condition follows similarly.

As was pointed out in class, this problem also admits a separated variables solution: $u(x,t) = A(x)B(t)$. Assuming this form, we find $AB' = A''B$ or $A''/A = B'/B$ Since the left expression only depends on x and the right expression depends only on t, it is easy to see that both quotients are independent of x and t, i.e., the common value of these quotients is a "separation constant" λ. (To see this just differentiate with respect to x or t.)

The boundary condition on the PDE then translates into $A(-1) = 0 = A(1)$. One can consider various cases to see that nonzero solutions of the resulting boundary value problem for A are only possible when $\lambda = -\omega^2$. Since we are only looking for a solution with a very special initial value, I can use a little foresight to simply take

$\lambda = -\pi^2$. Then one sees that $A(x) = \sin(\pi x)$ satisfies the ODE. Furthermore, the other equation $B' = -\pi^2 B$ leads to a solution of the PDE:

$$B(t) = ce^{-\pi^2 t} \qquad \text{and} \qquad u(x,t) = A(x)B(t) = e^{-\pi^2 t}\sin(\pi x).$$

Notice we have taken the integration constant $c = 1$ to satisfy the initial condition for the PDE.

By Theorem 5 on page 57 (the uniqueness theorem), the two solutions we have obtained are the same. It is not entirely obvious from the formulae. However, we note that

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{\eta\in\mathbb{R}} e^{-\frac{|\eta|^2}{4t}} \sin(\pi(x-\eta)) \\
&= \frac{1}{\sqrt{4\pi t}} \left[\int_{\eta\in\mathbb{R}} e^{-\frac{|\eta|^2}{4t}} \sin(\pi x)\cos(\eta) - \int_{\eta\in\mathbb{R}} e^{-\frac{|\eta|^2}{4t}} \cos(\pi x)\sin(\eta) \right] \\
&= \frac{1}{\sqrt{4\pi t}} \left[\sin(\pi x) \int_{\eta\in\mathbb{R}} e^{-\frac{|\eta|^2}{4t}} \cos(\eta) - \cos(\pi x) \int_{\eta\in\mathbb{R}} e^{-\frac{|\eta|^2}{4t}} \sin(\eta) \right] \\
&= \frac{\sin(\pi x)}{\sqrt{4\pi t}} \int_{\eta\in\mathbb{R}} e^{-\frac{|\eta|^2}{4t}} \cos(\eta).
\end{aligned}$$

Thus, the question is reduced to showing

$$\frac{1}{\sqrt{4\pi t}} \int_{\eta\in\mathbb{R}} e^{-\frac{|\eta|^2}{4t}} \cos(\eta) = e^{-\pi^2 t}.$$

Name and section: ____________________

2. (25 points) Find Duhamel's solution of

$$\begin{cases} u_t - \Delta u = t(1 - |x|)\chi_{[-1,1]}(x) \text{ on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = 0. \end{cases}$$

Verify that for each fixed t, your solution $u = u(x,t)$ has a unique maximum at $x = 0$.

Solution:

$$u(x,t) = \int_0^t \frac{1}{[4\pi(t-\tau)]^{n/2}} \left(\int_{\xi \in \mathbb{R}^n} e^{-\frac{|x-\xi|^2}{4(t-\tau)}} f(\xi, \tau) \right) d\tau.$$

As a first case, let us assume $0 \le x \le 1$.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \int_0^t \frac{1}{2[4\pi(t-\tau)]^{n/2}(t-\tau)} \left(\int_{\xi \in \mathbb{R}^n} (\xi - x) e^{-\frac{|x-\xi|^2}{4(t-\tau)}} f(\xi, \tau) \right) d\tau \qquad (1) \\
&= \int_0^t \frac{1}{2[4\pi(t-\tau)]^{n/2}(t-\tau)} \left(\int_{\eta \in B_1(-x)} \eta e^{-\frac{|\eta|^2}{4(t-\tau)}} f(x+\eta, \tau) \right) d\tau \\
&= \int_0^t \frac{1}{2[4\pi(t-\tau)]^{n/2}(t-\tau)} \left(\int_{-x-1}^0 \eta e^{-\frac{|\eta|^2}{4(t-\tau)}} f(x+\eta, \tau)\, d\eta \right. \\
&\qquad \left. + \int_0^{-x+1} \eta e^{-\frac{|\eta|^2}{4(t-\tau)}} f(x+\eta, \tau)\, d\eta \right) d\tau \\
&= \int_0^t \frac{1}{2[4\pi(t-\tau)]^{n/2}(t-\tau)} \left(-\int_0^{x+1} \xi e^{-\frac{|\xi|^2}{4(t-\tau)}} f(x-\xi, \tau)\, d\xi \right. \\
&\qquad \left. + \int_0^{-x+1} \eta e^{-\frac{|\eta|^2}{4(t-\tau)}} f(x+\eta, \tau)\, d\eta \right) d\tau \\
&= -\int_0^t \frac{1}{2[4\pi(t-\tau)]^{n/2}(t-\tau)} \left(\int_0^{1-x} \eta e^{-\frac{|\eta|^2}{4(t-\tau)}} [f(x-\eta, \tau) - f(x+\eta, \tau)]\, d\eta \right. \\
&\qquad \left. + \int_{1-x}^{x+1} \xi e^{-\frac{|\xi|^2}{4(t-\tau)}} f(x-\xi, \tau)\, d\xi \right) d\tau. \qquad (2)
\end{aligned}$$

Notice that if $0 < \eta \le 1 - x$, then $f(x-\eta, \tau) - f(x+\eta, \tau) = t(x + \eta - |x - \eta|) > 0$ unless $x = 0$. Furthermore, the ξ integral appearing in (2) is clearly nonnegative. Finally, if $0 < x \le 1$, one of the two integrals in the last expression above must always be positive for some nontrivial interval of the variable of integration. Owing to the negative sign, therefore, we have shown that $\partial u/\partial x \le 0$ when $0 \le x \le 1$ with equality holding and only holding for $x = 0$.

A similar string of inequalities shows that $\partial u/\partial x > 0$ for $-1 \le x < 0$. For $|x| \ge 1$, it follows directly from inspection of (1) that $\partial u/\partial x$ has the opposite sign of x. This shows that u is strictly increasing for $x < 0$, decreasing for $x > 0$, and has a unique maximum at $x = 0$ for each fixed time t.

Name and section: ____________________

Notice the same problem can be posed for $x \in \mathbb{R}^n$. It is a bit more technical to prove each "time slice" has a unique maximum at $x = 0$ in that case, but it's still true.

A nice alternative argument was given in the 1-D case by Gautam Goel: Starting from (1) separate the quantity $\partial u/\partial x$ into two terms of equal value. Use the change of variables $\eta = \xi - x$ in one of the terms as we have done. Use the change of variables $\eta = x - \xi$ in the other term. Then you get:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \int_0^t \frac{1}{4[4\pi(t-\tau)]^{n/2}(t-\tau)} \left(\int_{\eta\in\mathbb{R}^n} \eta e^{-\frac{|\eta|^2}{4(t-\tau)}} f(x+\eta,\tau) \right. \\ &\qquad \left. - \int_{\eta\in\mathbb{R}^n} \eta e^{-\frac{|\eta|^2}{4(t-\tau)}} f(x-\eta,\tau) \right) d\tau \\ &= \int_0^t \frac{1}{4[4\pi(t-\tau)]^{n/2}(t-\tau)} \left(\int_{\eta\in\mathbb{R}^n} \eta e^{-\frac{|\eta|^2}{4(t-\tau)}} [f(x+\eta,\tau) - f(x-\eta,\tau)] \right) d\tau\end{aligned} \tag{3}$$

From here, one checks by cases that $f(x+\eta,\tau) - f(x-\eta,\tau)$ has the same sign as η when $x < 0$ and the opposite sign when $x > 0$. Since strict inequality (positivity or negativity) of the integrand prevails unless $x = 0$, one obtains the same conclusion.

Name and section: ____________________

3. (25 points) (2.5.19) Solve the IVP

$$\begin{cases} v_{tt} = v_{xx} \text{ on } \mathbb{R} \times (0, \infty) \\ v(x,0) = 1/(1+x^2) \text{ on } \mathbb{R} \\ v_t(x,0) = 1 \text{ on } \mathbb{R}. \end{cases}$$

Solution: Set $w = v_t - v_x$. Then according to the equation, w should satisfy $w_t + w_x = 0$. Setting $\phi(s) = w(x+s, t+s)$, we find ϕ is a constant function of s. Thus, setting $s = -t$, we find $w(x-t, 0) = w(x,t)$, and hence

$$w(x,t) = 1 + \frac{2(x-t)}{[1+(x-t)^2]^2}.$$

We are thus confronted with the problem

$$v_t - v_x = 1 + \frac{2(x-t)}{[1+(x-t)^2]^2}.$$

with boundary value $v(x,0) = 1/(1+x^2)$. Setting $\psi(s) = v(x-s, t+s)$, we find

$$\psi'(s) = 1 - \frac{2(x-t-2s)}{[1+(x-t-2s)^2]^2}.$$

Integrating, we get

$$\begin{aligned} \psi(s) &= \psi(0) + s + \int_0^s \frac{2(x-t-2\sigma)}{[1+(x-t-2\sigma)^2]^2}\, d\sigma \\ &= \psi(0) + s + \frac{1}{2[1+(x-t-2\sigma)^2]}\Big|_{\sigma=0}^{s} \\ &= \psi(0) + s + \frac{1}{2[1+(x-t-2s)^2]} - \frac{1}{2[1+(x-t)^2]}. \end{aligned}$$

Thus, solving for $\psi(0)$ and evaluating at $s = -t$,

$$\begin{aligned} v(x,t) &= v(x+t, 0) + t - \frac{1}{2[1+(x+t)^2]} + \frac{1}{2[1+(x-t)^2]} \\ &= \frac{1}{1+(x+t)^2} + t - \frac{1}{2[1+(x+t)^2]} + \frac{1}{2[1+(x-t)^2]} \\ &= t + \frac{1}{2[1+(x+t)^2]^2} + \frac{1}{2[1+(x-t)^2]^2}. \end{aligned}$$

Name and section: ________________

4. (25 points) Find a solution of

$$\begin{cases} u_t - \Delta u = t \text{ on } [-1,1] \times (0,\infty) \\ u(-1,t) = 0, \ u(1,t) = 2. \end{cases}$$

Is your solutions unique? (Can you find another solution?)

Solution:

Though we didn't make specific mention of it, there were choices in the first problem about how to extend the initial values. We chose the extension, furthermore, so that a certain "balance" resulted in the boundary conditions $u(\pm 1, t) = 0$ being satisfied. We have no initial values in this problem, but we can try the same strategy and extend the forcing function in both *time and space* so that a similar balance prevails: Setting

$$f(x,t) = t \sum_{j \in \mathbb{Z}} (-1)^j \chi_{[2j-1,2j+1]}(x),$$

we consider the forced problem with a homogeneous initial condition:

$$\begin{cases} v_t - \Delta v = f \text{ on } \mathbb{R} \times (0,\infty) \\ v(x,0) = 0. \end{cases}$$

According to Theorem 2 on page 50, Duhamel's principle gives us a solution

$$\begin{aligned} v(x,t) &= \int_0^t \left(\int_{\xi \in \mathbb{R}^n} \Phi(x-\xi, t-\tau) f(\xi,\tau) \right) d\tau \\ &= \int_0^t \frac{\tau}{[4\pi(t-\tau)]^{n/2}} \sum_{j=-\infty}^{\infty} (-1)^j \int_{2j-1}^{2j+1} e^{-\frac{|x-\xi|^2}{4(t-\tau)}} \, d\xi \, d\tau. \end{aligned}$$

Using the same strategy suggested in problem 1, we next restrict this solution of the equation to $[-1,1]$ and attempt to verify the homogeneous boundary conditions $v(\pm 1, t) \equiv 0$. In fact, setting $x = -1$ and applying the change of variables $\eta = \xi + 1$, we find

$$\begin{aligned} v(x,t) &= \int_0^t \frac{\tau}{[4\pi(t-\tau)]^{n/2}} \sum_{j=-\infty}^{\infty} (-1)^j \int_{2j+1}^{2j+3} e^{-\frac{|\eta|^2}{4(t-\tau)}} \, d\eta \, d\tau \\ &= \int_0^t \frac{\tau}{[4\pi(t-\tau)]^{n/2}} \int_{\mathbb{R}} \sum_{j=-\infty}^{\infty} (-1)^j \chi_{[2j-1,2j+1]}(\eta - 1) e^{-\frac{|\eta|^2}{4(t-\tau)}} \, d\eta \, d\tau. \end{aligned}$$

Notice, however, that for fixed t and τ,

$$g(\eta) = \sum_{j=-\infty}^{\infty} (-1)^j \chi_{[2j-11,2j+1]}(\eta - 1) e^{-\frac{|\eta|^2}{4(t-\tau)}}$$

satisfies

$$\begin{aligned}
g(-\eta) &= \sum_{j=-\infty}^{\infty} (-1)^j \chi_{[2j-1,2j+1]}(-\eta-1) e^{-\frac{|\eta|^2}{4(t-\tau)}} \\
&= \sum_{j=-\infty}^{\infty} (-1)^j \chi_{[-2j-3,-2j+1]}(\eta-1) e^{-\frac{|\eta|^2}{4(t-\tau)}} \\
&= \sum_{k=-\infty}^{\infty} (-1)^{k-1} \chi_{[-2k-1,-2k+3]}(\eta-1) e^{-\frac{|\eta|^2}{4(t-\tau)}} \\
&= -\sum_{\ell=-\infty}^{\infty} (-1)^{-\ell} \chi_{[2\ell-1,2\ell+3]}(\eta-1) e^{-\frac{|\eta|^2}{4(t-\tau)}} \\
&= -g(\eta).
\end{aligned}$$

That is, g is odd, so it's integral is zero. Hence, $v(-1,t) \equiv 0$. (This is the consequence of the "balance" of temperature introduced in f.) The fact that $v(1,t) \equiv 0$ follows similarly.

It remains to modify v so that the boundary conditions $u(-1,t) = 0$ and $u(1,t) = 2$ are satisfied. For this we may use any solution $w = w(x)$ of the boundary value problem for Laplace's equation

$$\begin{cases} \Delta w = 0 \text{ on} \\ w(-1) = 0,\ w(1) = 2. \end{cases}$$

In one dimension $\Delta w = w''$ so this is easy to solve: $w(x) = x + 1$.

Thus, we have a solution for the original problem: $u = v + w$.

More generally, we could take $\tilde{w} = \tilde{w}(x,t)$ to be any solution of the homogeneous (unforced) problem

$$\begin{cases} \tilde{w}_t - \Delta\tilde{w} = 0 \text{ on } [-1,1] \times (0,\infty) \\ \tilde{w}(-1,0) = 0,\ \tilde{w}(1,0) = 2. \end{cases}$$

Writing $z = \tilde{w} - w$, we find that z can be any solution of

$$\begin{cases} z_t - \Delta z = 0 \text{ on } [-1,1] \times (0,\infty) \\ z(-1,0) = 0,\ z(1,0) = 0. \end{cases}$$

Thus, the question of uniqueness reduces to the uniqueness of the zero solution $z \equiv 0$ for this problem. However, we already constructed a nontrivial solution of this problem in problem 1 of this exam. Taking z to be that solution (or any number of others), we can then set $\tilde{w} = w + z$ and get a second distinct solution of the original problem. Since $v + w$ and $v + w + z$ are distinct solutions, uniqueness does not hold.