Math 6341, Exam 3: 3.2-4.1 (practice) Name and section: ________________

1. (25 points) (3.5.3) Show that

$$U(x, y, a, b) = \sqrt{1 - (x-a)^2 - (y-b)^2}$$

is a complete integral for the PDE

$$u^2(|Du|^2 + 1) = 1.$$

Find the corresponding envelope solution.

Solution: The first requirement is that for each a and b fixed $u(x, t) = U(x, y, a, b)$ solves the equation. In fact,

$$u_x = -\frac{x-a}{\sqrt{1-(x-a)^2-(y-b)^2}} \quad \text{and} \quad u_y = -\frac{y-b}{\sqrt{1-(x-a)^2-(y-b)^2}}.$$

Thus,

$$|Du|^2 + 1 = \frac{1}{1-(x-a)^2-(y-b)^2} = \frac{1}{u^2}.$$

Second, the 2×3 envelope matrix

$$\begin{pmatrix} \frac{x-a}{\sqrt{1-(x-a)^2-(y-b)^2}} & \frac{1-(y-b)^2}{(1-(x-a)^2-(y-b)^2)^{3/2}} & \frac{(x-a)(y-b)}{(1-(x-a)^2-(y-b)^2)^{3/2}} \\ \frac{y-b}{\sqrt{1-(x-a)^2-(y-b)^2}} & \frac{1-(x-a)^2}{(1-(x-a)^2-(y-b)^2)^{3/2}} & \frac{(x-a)(y-b)}{(1-(x-a)^2-(y-b)^2)^{3/2}} \end{pmatrix}.$$

is required to have rank 2. Notice that the determinant of the 2×2 matrix obtained by deleting the last column is

$$\frac{x-a-(x-a)^3-(y-b)+(y-b)^3}{(1-(x-a)^2-(y-b)^2)^2}.$$

This is nonzero unless $(x-a)[1-(x-a)^2] = (y-b)[1-(y-b)^2]$. Since for fixed x and y, the set of $(a, b) \in \mathbb{R}^2$ satisfying this relation is a closed set, we obtain an open set (the complement) on which the envelope matrix has rank at least 2 (and it can't have rank more than 2), so

$$U(x, y, a, b) = \sqrt{1-(x-a)^2-(y-b)^2}$$

provides a complete integral on that set.

To obtain the envelope, we consider the system of equations $U_a = 0$ and $U_b = 0$ and attempt to solve for a and b: These equations are

$$\frac{x-a}{\sqrt{1-(x-a)^2-(y-b)^2}} = 0 \quad \text{and} \quad \frac{y-b}{\sqrt{1-(x-a)^2-(y-b)^2}} = 0.$$

The solution is $a = x$ and $b = y$. Thus, the envelope is

$$U(x, y, x, y) = 1.$$

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2. (25 points) (3.5.5) Use the method of characteristics to solve

$$\begin{cases} xu_x + yu_y = 2u \text{ on } \mathbb{R}^2 \\ u(x,1) = x. \end{cases}$$

Solution: We set $v(t) = u(\xi(t), \eta(t))$ where (ξ, η) parameterizes a curve passing through (x, y) and satisfying $\xi' = \xi$, $\eta' = \eta$.

In this way, we see that $\xi = xe^t$, $\eta = ye^t$, and

$$v' = xe^t u_x(xe^t, ye^t) + ye^t u_y(xe^t, ye^t) = 2v.$$

It follows that $v = v(0)e^{2t}$.

Next, for fixed (x, y), we wish to find t such that $\eta(t) = 1$ (in order to hit the Cauchy data curve). That is, we want $ye^t = 1$ or $e^t = 1/y$ (at least away from $y = 0$.

Substituting this choice into the expression for v, we find

$$u(x/y, 1) = u(x, y)/y^2.$$

That is,

$$u(x, y) = xy$$

which is easily seen to be an entire solution of the PDE.

3. (25 points) (3.5.20) Find the characteristic curves and the solution of the IVP

$$\begin{cases} u_t + (u^2/2)_x = 0 \text{ on } \mathbb{R} \times (0, \infty) \\ \\ u(x,0) = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 1, & 1 \leq x. \end{cases} \end{cases}$$

Solution: We first attempt to find characteristic curves. Setting $v = u(\xi, \tau)$, we want

$$\xi' = v, \quad \tau' = 1,$$

and

$$v' = \xi' u_x(\xi, \tau) + \tau' u_t(\xi, \tau) = 0.$$

Solving this system of ODEs, we get $v = u(x_0)$, assuming we cross the Cauchy data curve $t = 0$ at a point x_0,

$$\xi = u(x_0)t + x_0,$$

and $\tau = t$. Evidently, there are three cases:

1. $x_0 \leq 0$. Then the characteristics $\xi \equiv x_0$, $\tau = t$ cover the region $x \leq 0$ and yield the solution
$$u \equiv 0, \quad x \leq 0.$$

2. $0 \leq x_0 \leq 1$. Here the characteristics are
$$\xi = x_0 t + x_0 = x_0(t+1), \quad \tau = t.$$
Such a characteristic hits the point (x,t) if $0 \leq x \leq 1+t$ and
$$x_0 = x/(1+t).$$
It follows that
$$u(x,t) = \frac{x}{1+t}, \quad 0 \leq x \leq 1+t.$$

3. The remaining characteristics with $x_0 \geq 1$ are given by $\xi = t + x_0$. These cover the region $x \geq 1+t$, and yield the solution $u \equiv 1$ there.

4. (25 points) (4.7.2) Find a separated variables solution of
$$\begin{cases} \Delta u = 0 \text{ on } \mathbb{R}^2 \\ u(x,0) = 0, \quad u_y(x,0) = \sin x. \end{cases}$$
Explain your reasoning carefully.

Solution: Setting $u = f(x)g(y)$, we obtain away from $f = 0$ or $g = 0$ a separation constant λ such that
$$-\frac{f''}{f} = \frac{g''}{g} = \lambda.$$
Thus, we obtain two ODEs
$$f'' = -\lambda f \quad \text{and} \quad g'' = \lambda g.$$
The first boundary condition gives $f(x)g(0) = 0$ from which we conclude $g(0) = 0$, since $f(x) = 0$ cannot lead to a solution satisfying the other boundary condition.

The second boundary condition is $f(x)g'(0) = \sin x$. Therefore,
$$f(x) = \frac{\sin x}{g'(0)}.$$
It follows from this that $f'' = -f$. In view of the first ODE, we must have $\lambda = 1$. The second ODE with the first boundary condition then yeilds
$$g(y) = g'(0)\sinh y.$$
The solution is thus,
$$u(x,y) = f(x)g(y) = \sinh y \sin x.$$