Math 6341, Exam 2: 2.2-2.3 (practice) Name and section: ____________________

1. (25 points) (2.5.7,11) Solve the boundary value problem (BVP)

$$\begin{cases} \Delta u = 0 \text{ on } B_2(0,0) \subset \mathbb{R}^2 \\ u(x,y)_{(x,y)\in\partial B_2(0,0)} = -\ln\sqrt{x^2+(y-1)^2}\Big/(2\pi). \end{cases}$$

Solution: The boundary values are those of the fundamental solution centered at $(0,1)$. The inversion (Kelvin transform) of this point is $(0,4)$, and we remember that the solution of this problem can be constructed from the fundamental solution with singularity at $(0,4)$:

$$\Phi(x,y-4) = -\frac{1}{2\pi}\ln\sqrt{x^2+(y-4)^2}.$$

This is harmonic in the ball, but the problem is that the boundary values are not the same. Notice, however that for $x^2+y^2=4$ we have

$$x^2+(y-1)^2 = 4-2y+1$$

and

$$x^2+(y-4)^2 = 4-8y+16 = 4(1-2y+4).$$

Thus, if we scale the argument of the square root by $1/4$, then the harmonicity will be preserved and the boundary values will also be correct:

$$u(x,y) = -\frac{1}{2\pi}\ln\sqrt{[x^2+(y-4)^2]/4}.$$

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2. (25 points) (Green's Identities) If $u, v \in C^2(\bar{\Omega})$ satisfy

$$\begin{cases} \Delta u = f \\ u\big|_{\partial\Omega} = u_0, \end{cases}$$

and

$$\begin{cases} \Delta v = 0 \\ v(\xi)\big|_{\xi\in\partial\Omega} = \Phi(\xi - x)\big|_{\xi\in\partial\Omega}, \end{cases}$$

where Φ is the fundamental solution of Laplace's PDE, then find a formula expressing

$$\int_{\xi\in\partial\Omega} \Phi(\xi - x) Du(\xi) \cdot n$$

where n is the outward normal to $\partial\Omega$ in terms of f, u_0, v, and Dv.

Solution: By Green's second identity

$$\int_\Omega (u\Delta v - v\Delta u) = \int_{\partial\Omega} (uDv \cdot n - vDu \cdot n).$$

Substituting from the boundary value problems, this becomes

$$-\int_\Omega vf = \int_{\partial\Omega} u_0 Dv \cdot n - \int_{\xi\in\partial\Omega} \Phi(\xi - x) Du(\xi) \cdot n.$$

Thus,

$$\int_{\xi\in\partial\Omega} \Phi(\xi - x) Du(\xi) \cdot n = \int_\Omega vf + \int_{\partial\Omega} u_0 Dv \cdot n.$$

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3. (25 points) (2.5.12) If u is a solution of the heat equation on $\mathbb{R}^n \times (0, \infty)$, show that

$$v(x,t) = x \cdot Du(x,t) + 2tu_t(x,t)$$

is also a solution.

Solution:

$$v_t = x \cdot Du_t(x,t) + 2u_t(x,t) + 2tu_{tt}(x,t)$$

$$v_{x_j} = D_{x_j}u(x,t) + x \cdot DD_{x_j}u(x,t) + 2tu_{x_j t}(x,t)$$

$$\Delta v = \Delta u + \Delta u + x \cdot D(\Delta u) + 2t(\Delta u)_t.$$

$$v_t - \Delta v = x \cdot D(u_t - \Delta u) + 2u_t - 2\Delta u + 2t(u_t - \Delta u)_t = 0.$$

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4. (25 points) (2.5.14) Solve the initial value problem (IVP)

$$\begin{cases} u_t = \Delta u + 5u \text{ on } \mathbb{R}^2 \times (0,\infty) \\ u(x,y,0) = 1/(1+x^2+y^2) \text{ on } \mathbb{R}^2. \end{cases}$$

Solution: Multiply through the equation by e^{-5t} to obtain

$$e^{-5t}u_t - 5e^{-5t}u = (e^{-5t}u)_t = \Delta(e^{-5t}u).$$

That is, $u = e^{5t}v$ where v satisfies

$$\begin{cases} v_t = \Delta v \text{ on } \mathbb{R}^2 \times (0,\infty) \\ v(x,y,0) = 1/(1+x^2+y^2) \text{ on } \mathbb{R}^2. \end{cases}$$

This equation can be solved by convolution with the fundamental solution

$$\Phi(x,y,t) = \frac{1}{4\pi t} e^{-\frac{x^2+y^2}{4t}}.$$

In fact,

$$\begin{aligned} v(x,y,t) &= \int_{\mathbb{R}^2} \frac{1}{4\pi t(1+\xi^2+\eta^2)} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4t}} \\ &= \int_0^{2\pi}\int_0^\infty \frac{1}{4\pi t(1+r^2)} e^{-\frac{(x-r\sin\theta)^2+(y-r\theta)^2}{4t}}\, r\,dr\,d\theta. \end{aligned}$$

Thus,

$$u(x,y,t) = \frac{e^{5t}}{4\pi t}\int_0^{2\pi}\int_0^\infty \frac{1}{1+r^2} e^{-\frac{(x-r\cos\theta)^2+(y-r\sin\theta)^2}{4t}}\, r\,dr\,d\theta.$$