**Math 6341, Exam 2: 2.3-3.2 (practice)** Name and section: ____________________

1. (25 points) (2.5.14) Solve the initial value problem (IVP)

$$\begin{cases} u_t = \Delta u + 5u \text{ on } \mathbb{R}^2 \times (0, \infty) \\ u(x, y, 0) = 1/(1 + x^2 + y^2) \text{ on } \mathbb{R}^2. \end{cases}$$

> **Solution:** Multiply through the equation by $e^{-5t}$ to obtain
>
> $$e^{-5t}u_t - 5e^{-5t}u = (e^{-5t}u)_t = \Delta(e^{-5t}u).$$
>
> That is, $u = e^{5t}v$ where $v$ satisfies
>
> $$\begin{cases} v_t = \Delta v \text{ on } \mathbb{R}^2 \times (0, \infty) \\ v(x, y, 0) = 1/(1 + x^2 + y^2) \text{ on } \mathbb{R}^2. \end{cases}$$
>
> This equation can be solved by convolution with the fundamental solution
>
> $$\Phi(x, y, t) = \frac{1}{4\pi t} e^{-\frac{x^2+y^2}{4t}}.$$
>
> In fact,
>
> $$\begin{aligned} v(x, y, t) &= \int_{\mathbb{R}^2} \frac{1}{4\pi t(1 + \xi^2 + \eta^2)} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4t}} \\ &= \int_0^{2\pi} \int_0^\infty \frac{1}{4\pi t(1 + r^2)} e^{-\frac{(x-r\sin\theta)^2+(y-r\theta)^2}{4t}} \, r \, dr \, d\theta. \end{aligned}$$
>
> Thus,
>
> $$u(x, y, t) = \frac{e^{5t}}{4\pi t} \int_0^{2\pi} \int_0^\infty \frac{1}{1 + r^2} e^{-\frac{(x-r\cos\theta)^2+(y-r\sin\theta)^2}{4t}} \, r \, dr \, d\theta.$$

2. (25 points) (2.5.19) Show that every solution of the equation $u_{xy} = 0$ on all of $\mathbb{R}^2$ has the form $u(x, y) = F(x) + G(y)$.

> **Solution:** Assuming we have a classical $C^2$ solution $u$, we can imagine that $x$ is fixed and integrate the equation from 0 to $y$ to obtain
>
> $$u_x(x, y) = u(x, 0) = f(x).$$
>
> Since $x$ was arbitrary, this gives an identity in $x$ and $y$. That is, $u$ satisfies $u_x = f$. Integrating next from 0 to $x$ (with $y$ fixed), we find
>
> $$u(x, y) = u(0, y) + \int_0^x f(\xi) d\xi.$$

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Since $f$ was a $C^1$ function, we see that $F(x) = \int_0^x f(\xi)d\xi$ is $C^2$. Setting $G(y) = u(0,y)$, we have shown that $u(x,y) = F(x) + G(y)$ for some $C^2$ functions $F$ and $G$.

3. (25 points) (2.5.19) Solve the IVP

$$\begin{cases} v_{tt} = v_{xx} \text{ on } \mathbb{R} \times (0,\infty) \\ v(x,0) = 1/(1+x^2) \text{ on } \mathbb{R} \\ v_t(x,0) = 1 \text{ on } \mathbb{R}. \end{cases}$$

**Solution:** Set $w = v_t - v_x$. Then according to the equation, $w$ should satisfy $w_t + w_x = 0$. Setting $\phi(s) = w(x+s, t+s)$, we find $\phi$ is a constant function of $s$. Thus, setting $s = -t$, we find $w(x-t, 0) = w(x,t)$, and hence

$$w(x,t) = 1 + \frac{2(x-t)}{[1+(x-t)^2]^2}.$$

We are thus confronted with the problem

$$v_t - v_x = 1 + \frac{2(x-t)}{[1+(x-t)^2]^2}.$$

with boundary value $v(x,0) = 1/(1+x^2)$. Setting $\psi(s) = v(x-s, t+s)$, we find

$$\psi'(s) = 1 - \frac{2(x-t-2s)}{[1+(x-t-2s)^2]^2}.$$

Integrating, we get

$$\begin{aligned} \psi(s) &= \psi(0) + s + \int_0^s \frac{2(x-t-2\sigma)}{[1+(x-t-2\sigma)^2]^2}\, d\sigma \\ &= \psi(0) + s + \frac{1}{2[1+(x-t-2\sigma)^2]}\Big|_{\sigma=0}^{s} \\ &= \psi(0) + s + \frac{1}{2[1+(x-t-2s)^2]} - \frac{1}{2[1+(x-t)^2]}. \end{aligned}$$

Thus, solving for $\psi(0)$ and evaluating at $s = -t$,

$$\begin{aligned} v(x,t) &= v(x+t,0) + t - \frac{1}{2[1+(x+t)^2]} + \frac{1}{2[1+(x-t)^2]} \\ &= \frac{1}{1+(x+t)^2} + t - \frac{1}{2[1+(x+t)^2]} + \frac{1}{2[1+(x-t)^2]} \\ &= t + \frac{1}{2[1+(x+t)^2]^2} + \frac{1}{2[1+(x-t)^2]^2}. \end{aligned}$$

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4. (25 points) (3.5.1) Show that

$$U(x,t,\xi,s) = \xi \cdot x - tH(\xi) + s$$

is a complete integral for the Hamilton-Jacobi equation

$$u_t + H(Du) = 0.$$

**Solution:** The first requirement is that for each $\xi$ and $s$ fixed $u(x,t) = U(x,t,\xi,s)$ solves the equation. In fact,

$$u_t = -H(\xi)$$

and

$$Du = \xi.$$

Thus, $u_t + H(Du) = -H(\xi) + H(\xi) = 0$. So we have a family of solutions.

Second, the $(n+1) \times (n+2)$ envelope matrix

$$\begin{pmatrix} x_1 - tH_1(\xi) & 1 & 0 & & 0 & -H_1(\xi) \\ x_2 - tH_2(\xi) & 0 & 1 & & 0 & -H_2(\xi) \\ \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ x_n - tH_n(\xi) & 0 & & & 1 & -H_n(\xi) \\ 1 & 0 & \dots & & \dots & 0 \end{pmatrix}$$

is required to have rank $n+1$. Since rank is preserved by elementary row operations, we may use the last row to eliminate all entries in the first column except the last one. Then moving the last row from the bottom to the top, we get the $(n+1)\times(n+1)$ identity with one additional column appended. This clearly has rank $n+1$. Thus,

$$U(x,t,\xi,s) = \xi \cdot x - tH(\xi) + s$$

provides a complete integral.