Math 6341, Exam 1: 2.1-3 (practice) Name and section: ____________________

1. (25 points) (2.5.1) Find $u : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ if

$$\begin{cases} u_t + (3,5) \cdot Du + 4u = 0 \text{ on } (0, \infty) \\ u(x, y, 0) = \phi(x^2 + y^2), \end{cases}$$

where ϕ is the standard compactly supported function on $\mathbb{R}$ with $\phi(s) = \exp(1/(|s| - 1))$ for $|s| < 1$.

Solution: We fix $x \in \mathbb{R}^2$ and t and set $v(s) = u(x + s(3, 5), t + s)$. Then we find

$$v'(s) = -4v(s).$$

It follows that $v(s) = v(0)e^{-4s}$. Translating this back into terms of u, we have

$$u(x + s(3, 5), t + s) = u(x, t)e^{-4s}. \tag{1}$$

Perhaps the most obvious thing to write down next (since we're looking for $u(x, t)$) is

$$u(x, t) = u(x + s(3, 5), t + s)e^{4s}.$$

Of course, it looks like there is a bit of a problem with this since the right side depends on the unknown u. However, this holds for all s, and if we could choose s to make the second argument of u on the right vanish, then we could use the boundary condition to eliminate that u dependence. That is, if we take $s = -t$, then

$$u(x, t) = u(x - t(3, 5), 0)e^{-4t} = \phi((x - 3t)^2 + (y - 5t)^2)e^{-4t}.$$

And that is a formula for the solution.

An alternative way to look at the last part of this reasoning is the following. Set $t = 0$ in (1). Then you get

$$u(x + s(3, 5), t) = \phi(x_1^2 + x_2^2)e^{-4s}.$$

This looks almost like a formula for $u(x, t)$, but the first argument is not really x. Rename the first argument something like $\xi = (\xi_1, \xi_2)$. Then go back and solve for x_1 and x_2 in terms of ξ and s. Then you get a formula for $u(\xi, s)$. Since you want u and don't really care about the variable names, this works just fine.

2. (25 points) (2.5.2) If $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic, show that $v(x, y) = u \circ f(x, y)$ is also harmonic if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear orthogonal map.

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Solution:

$$Dv = (Du \circ f)A$$

where $A = Df$ is the matrix associated with f. Also,

$$D^2v = A^T(D^2u \circ f)A.$$

Thus, $\Delta v = \operatorname{trace} A^t(D^2u \circ f)A$.

On the other hand, A is a matrix of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where $a^2 + b^2 = 1$. Thus, computing directly we find

$$\begin{aligned} \Delta v &= \operatorname{trace} \begin{pmatrix} a^2D_{11}u + abD_{12}u + abD_{21}u + b^2D_{22}u & * \\ * & b^2D_{11}u - abD_{12}u - abD_{21}u + a^2D_{22}u \end{pmatrix} \\ &= (a^2 + b^2)D_{11}u + (b^2 + a^2)D_{22}u \\ &= \Delta u \\ &= 0. \end{aligned}$$

A more sophisticated way to look at this (which also works better for the same question in general dimensions) is the following: Start with an arbitrary domain Ω in $\mathbb{R}^2$.

$$\begin{aligned} \int_\Omega \operatorname{div} Dv &= \int_{\partial\Omega} Dv \cdot n \\ &= \int_{f(\partial\Omega)} (Dv \circ f^{-1}) \cdot (n \circ f^{-1}) \\ &= \int_{\partial f(\Omega)} DuA \cdot (n \circ f^{-1}) \\ &= \int_{\partial f(\Omega)} Du \cdot A(n \circ f^{-1}) \\ &= \int_{\partial f(\Omega)} Du \cdot \nu \\ &= \int_{f(\Omega)} \operatorname{div} Du \\ &= 0. \end{aligned}$$

The first and sixth identities are the divergence theorem. The second identity is a change of variables formula. In the fifth identity, ν is the unit outward normal to $f(\Omega)$ which transforms by the rotation. Finally, since this computation holds for arbitrary Ω, we get $\Delta v = 0$.

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3. (25 points) (2.5.5) If $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and convex, then show that $v = \phi \circ u$ is subharmonic.

Solution: Subharmonic means that the Laplacian is non-negative. Thus, we need to show $\Delta v \geq 0$. We compute:

$$D_j v = (\phi' \circ u) D_j u$$

and

$$D_{jj} v = (\phi'' \circ u)(D_j u)^2 + (\phi' \circ u) D_{jj} u.$$

Thus,

$$\begin{aligned}
\Delta v &= \sum (\phi'' \circ u)(D_j u)^2 + (\phi' \circ u) D_{jj} u \\
&= (\phi'' \circ u)|Du|^2 + (\phi' \circ u)\Delta u \\
&= (\phi'' \circ u)|Du|^2.
\end{aligned}$$

Since ϕ is convex, we know that $\phi'' \geq 0$. It follows that $\Delta v \geq 0$ as desired.

4. (25 points) (2.5.12) If u is a smooth solution of the heat equation on $\mathbb{R}^n \times (0, \infty)$, show that

$$u_\lambda(x, t) = u(\lambda x, \lambda^2 t)$$

is also a solution of the heat equation for any $\lambda \in \mathbb{R}$.

Solution: Recall that the heat equation has the form $u_t = \Delta u$.

Let $v = u_\lambda$. Then $v_t(x, t) = \lambda^2 u_t(\lambda x, \lambda^2 t)$ by the chain rule. Similarly, $D_j v = \lambda D_j u(\lambda x, \lambda^2 t)$, and $D_{jj} v = \lambda^2 D_{jj} u(\lambda x, \lambda^2 t)$. Thus,

$$\begin{aligned}
v_t &= \lambda^2 \sum D_{jj} u(\lambda x, \lambda^2 t) \\
&= \sum \lambda^2 D_{jj} u(\lambda x, \lambda^2 t) \\
&= \sum D_{jj} v \\
&= \Delta v.
\end{aligned}$$