Mean Value Formulae for Laplace and Heat Equation

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Abstract

Here I discuss a method to construct the mean value theorem for the heat equation. To construct such a formula *ab initio*, I first generalize the method used in the mean-value theorem for the Laplace equation which is discussed. Most of the generalization comes from using the co-area formula and its modified version. I found out that a mean value formula on the surface is also possible for the heat equation as in the Laplace Equation. Both these forms and many other mean value formulae are possible using a general mean value formula for each of the Laplace and Heat equations.

1 Co-area Formula

I will be using the Coarea Formula given in the book (C-2 Theorem 5) and a modified version of it as below. Following the book, the coarea formula is:

$$\int_{\mathbb{R}^n} u|Dv|\,\mathrm{d}x \quad = \quad \int_{-\infty}^{\infty}\Big(\int_{v\equiv r} u\,\mathrm{d}s\Big)\,\mathrm{d}r$$

Let us define $U(r)$ and $\partial U(r)$ as:

$$\begin{aligned} U(r) &:= \{x \in \mathbb{R}^n \mid v(x) \leq r\} \\ \partial U(r) &:= \{x \in \mathbb{R}^n \mid v(x) = r\} \end{aligned}$$

I assume that v a function such that the set $U(r)$ always decreases in size with r. Remember that r is not the radius but the value of the function v on the level set.

Using $U(r)$, I propose

$$\int_{U(R)} u|Dv|\,\mathrm{d}x = \int_R^\infty \left(\int_{\partial U(r)} u\,\mathrm{d}s \right) \mathrm{d}r \tag{1.1a}$$

$$\frac{\partial}{\partial R} \int_{U(R)} u|Dv|\,\mathrm{d}x = - \int_{\partial U(r)} u\,\mathrm{d}s \tag{1.1b}$$

$$\int_{U(R)} u\,\mathrm{d}x = \int_R^\infty \left(\int_{\partial U(r)} \frac{u}{|Dv|}\,\mathrm{d}s \right) \mathrm{d}r \tag{1.1c}$$

$$\frac{\partial}{\partial R} \int_{U(R)} u\,\mathrm{d}x = - \left(\int_{\partial U(r)} \frac{u}{|Dv|}\,\mathrm{d}s \right) \mathrm{d}r \tag{1.1d}$$

1.1 A note

I will encounter such equation but with $D_x v$ instead of Dv. For that,

$$\begin{aligned}
\int_{U(R)} u\,\mathrm{d}x &= \int_{T_1(R)}^{T_2(R)} \left(\int_{U_x(R,t)} u\,\mathrm{dx} \right) \mathrm{dt} \\
&= \int_{T_1(R)}^{T_2(R)} \left(\int_R^\infty \int_{\partial U_x(r,t)} \frac{u}{|D_x v|}\,\mathrm{dx}\,\mathrm{dr} \right) \mathrm{dt} \\
&= \int_R^\infty \int_{T_1(R)}^{T_2(R)} \int_{\partial U_x(r,t)} \frac{u}{|D_x v|}\,\mathrm{dx}\,\mathrm{dt}\,\mathrm{dr}
\end{aligned}$$

Note that the integral $\int_{T_1(R)}^{T_2(R)}$ can be replaced by $\int_{T_1(r)}^{T_2(r)}$ as in our case, $\frac{\partial T_1(r)}{\partial r} \geq 0$ and $\frac{\partial T_2(r)}{\partial r} \leq 0$, due to the kinds of v we are considering. This gives us,

$$\begin{aligned}
\int_{U(R)} u\,\mathrm{d}x &= \int_R^\infty \int_{T_1(r)}^{T_2(r)} \int_{\partial U_x(r,t)} \frac{u}{|D_x v|}\,\mathrm{dx}\,\mathrm{dt}\,\mathrm{dr} \\
&= \int_R^\infty \left(\int_{\partial U(r)} \frac{u}{|D_x v|}\,\mathrm{ds} \right) \mathrm{dr} \qquad (1.2)
\end{aligned}$$

This is strange as (1.2) is same as (1.1c) with only $|Dv|$ replaced by $|D_x v|$. I don't have a good understanding why this is so, but I tried it for calculating the volume of a sphere and it works! I am not sure but this has something to do with the integral $\int_{\partial U_x(r,t)} \frac{u}{|D_x v|}\,\mathrm{dx}$ being 0 at the ends, i.e. at $T_1(r)$ and $T_2(r)$. The differential form of equation (1.2) will be,

$$\frac{\partial}{\partial R} \int_{U(R)} u\,\mathrm{d}x = - \left(\int_{\partial U(r)} \frac{u}{|D_x v|}\,\mathrm{ds} \right) \mathrm{d}r \tag{1.3}$$

$$\textit{and}\ \frac{\partial}{\partial R} \int_{U(R)} u|D_x v|\,\mathrm{d}x = - \left(\int_{\partial U(r)} u\,\mathrm{d}s \right) \mathrm{d}r \tag{1.4}$$

2 Laplace Equation

2.1 General case of v

I will try to generalize the the mean value formula using the co-area formula discussed above.

$$\begin{aligned}0 &= \Delta u \\ &= \int_{U(r)} \Delta u \\ &= \int_{\partial U(r)} \vec{Du} \cdot \vec{n}\end{aligned}$$

The surface normal $\vec{n} = \frac{\vec{Dv}}{|Dv|}$. Now using 1.1d

$$\begin{aligned}0 &= \int_{\partial U(r)} \frac{\vec{Du} \cdot \vec{Dv}}{|Dv|} \\ &= \frac{\partial}{\partial r} \int_{U(r)} \vec{Du} \cdot \vec{Dv}\end{aligned}$$

Using Green's Formula,

$$\begin{aligned}0 &= -\frac{\partial}{\partial r} \int_{U(r)} u \Delta v + \frac{\partial}{\partial r} \int_{\partial U(r)} u \frac{\vec{Dv} \cdot \vec{Dv}}{|Dv|} \\ &= -\frac{\partial}{\partial r} \int_{U(r)} u \Delta v + \frac{\partial}{\partial r} \int_{\partial U(r)} u|Dv| \qquad (2.1)\end{aligned}$$

This gives a hint of using v which satisfies the laplace equation and makes the first integral 0. In case of the fundamental solution ϕ or $\phi(x - x_0; t - t_0)$, $\Delta\phi$ is a δ-distribution, which gives,

$$\frac{\partial}{\partial r} \int_{U(r)} u \Delta_x \phi = \frac{\partial}{\partial r} u(x_0) = 0$$

Hence,

$$\begin{aligned}\frac{\partial}{\partial r} \int_{\partial U(r)} u|Dv| &= 0 \\ or \int_{\partial U(r)} u|Dv| &= constant \\ &:= k(v) \qquad (2.2)\end{aligned}$$

$k(v)$ can be calculated by finding the limiting value as $r \to \infty$,

$$k(v) = \lim_{r\to\infty} \int_{\partial U(r)} u|Dv|$$

Equation (2.2) holds for any v that satisfies the heat equation, or like ϕ acts as a δ distribution.

2.2 Special case of ϕ

Let's proceed with $v = \phi(x - x_0)$. I'll use x for $x - x_0$ and ϕ for $\phi(x - x_0)$.

$$\begin{aligned} k(\phi) &= \lim_{r\to\infty} \int_{\partial U(r)} u|D\phi| \\ &= u(x_0) \end{aligned}$$

Hence,

$$\begin{aligned} u(x_0) &= \int_{\partial U(r)} u|D\phi| \\ &= |D\phi| \int_{\partial U(r)} u \\ &= \frac{1}{n\alpha(n)|x - x_0|^{n-1}} \int_{\partial U(r)} u \\ \therefore u(x_0) &= \fint_{\partial U(r)} u \end{aligned} \tag{2.3}$$

which is the mean value theorem for the Laplace equation. The other mean value theorem can be derived from 2.2 as follows. I couldn't generalize this part more and had to use a ϕspecific method. Using the fact that $|D\phi|$ is constant on a level set, multiply both sides of 2.3 by $\frac{1}{|D\phi|^2}$.

$$\int_{\partial U(r)} \frac{u}{|D\phi|} = u(x_0)\frac{1}{|D\phi|^2}$$

Now, integrating from R to ∞, we get

$$\begin{aligned} \int_R^\infty \int_{\partial U(r)} \frac{u}{|D\phi|} &= u(x_0) \int_R^\infty \frac{1}{|D\phi|^2} \\ \Rightarrow \int_{U(r)} u &= (\alpha(n)|x - x_0|^n)\, u(x_0) \\ \therefore \fint_{U(r)} u &= u(x_0) \end{aligned} \tag{2.4}$$

Equations 2.4 and 2.3 are the mean value formulae pair for the Laplace Equation. Both of these and other mean value formulas are possible from a general mean value formula on the surface of a level set v given by 2.2. An example of another mean value formula possible is as follows.

2.3 Example of another mean value formula

I will start with equation 2.3 and instead of multiplying by $\frac{1}{|D\phi|^2}$, it is multiplied by $\frac{1}{\phi^2}$ (which is same as $\frac{1}{r^2}$ on the surface) and then integrate.

$$
\begin{aligned}
u(x_0) &= \int_{\partial U(r)} u|D\phi| \\
u(x_0)\int_R^\infty \frac{1}{r^2} &= \int_R^\infty \int_{\partial U(r)} u\frac{|D\phi|}{\phi^2} \\
\frac{u(x_0)}{R} &= (n-2)^2 \int_{U(R)} \frac{u}{|x|^2} \\
\therefore u(x_0) &= R(n-2)^2 \int_{U(R)} \frac{u}{|x|^2}
\end{aligned}
$$

Here, R is the value of ϕ at the surface. replacing it in terms of r (radius of the ball), we get

$$
u(x_0) = \frac{(n-2)r^2}{n} \fint_{B_r(x_0)} \frac{u}{|x|^2}
$$

3 Heat Equation

3.1 General case of v

I try to follow the same procedure as above for the Heat Equation.

$$
\begin{aligned}
u_t &= -\Delta_x u \\
\int_U u_t &= -\int_U \Delta_x u \\
\int_{\partial U} un_t &= -\int_{\partial U} \vec{D_x} u \cdot \vec{n}
\end{aligned}
$$

The surface normals are characterized by $\vec{n} = \frac{\vec{Dv}}{|Dv|}$, $\vec{n_x} = \frac{\vec{D_x v}}{|D_x v|}$ and $n_t = \vec{n} \cdot \hat{t} = \frac{v_t}{|Dv|}$. Hence,

$$
\int_{\partial U(r)} u\frac{v_t}{|Dv|} = -\int_{\partial U(r)} \frac{\vec{D_x} u \cdot \vec{D_x} v}{|D_x v|}
$$

Using (1.3),

$$
\frac{\partial}{\partial r}\int_{U(r)} uv_t = \frac{\partial}{\partial r}\int_{U(r)} \vec{D_x} u \cdot \vec{D_x} v
$$

Using Green's Formula,

$$
\begin{aligned}
\frac{\partial}{\partial r}\int_{U(r)} uv_t &= \frac{\partial}{\partial r}\int_{T_1}^{T_2}\left(-\int_{U_x(r,t)} u\Delta_x v + \int_{\partial U_x(r,t)} u\frac{\vec{D_x} v \cdot \vec{D_x} v}{|D_x v|}\right) \\
&= -\frac{\partial}{\partial r}\left(\int_{U(r)} u\Delta_x v\right) + \frac{\partial}{\partial r}\left(\int_{\partial U(r)} u|D_x v|\right)
\end{aligned}
$$

This gives the following equation which is similar to (2.1).

$$\therefore 0 = -\frac{\partial}{\partial r}\int_{U(r)} u\left(v_t + \Delta_x v\right) + \frac{\partial}{\partial r}\int_{\partial U(r)} u|D_x v| \tag{3.1}$$

Similar arguments hold as before. I choose v that satisfies the heat equation and makes the first integral 0. In case of the fundamental solution ϕ or $\phi(x - x_0; t - t_0)$, $\phi_t + \Delta_x \phi$ is a δ distribution which gives,

$$\frac{\partial}{\partial r}\int_{U(r)} u\left(\phi_t + \Delta_x \phi\right) = \frac{\partial}{\partial r} u(x_0; t_0) = 0$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial r}\int_{\partial U(r)} u|D_x v| &= 0 \\ or \int_{\partial U(r)} u|D_x v| &= constant \\ &:= k_v \end{aligned} \tag{3.2}$$

$k(v)$ can be calculated by finding the limiting value as $r \to \infty$,

$$k_v = \lim_{r\to\infty}\int_{\partial U(r)} u|D_x v|$$

Equation (3.2) holds for any v that satisfies the heat equation, or like ϕ acts as a δ distribution.

3.2 Special case of ϕ

Let's proceed with $v = \phi(x - x_0; t - t_0)$. I'll use x for $x - x_0$, t for $t - t_0$ and ϕ for $\phi(x - x_0)$.

$$\begin{aligned} k_\phi &= \lim_{r\to\infty}\int_{\partial U(r)} u|D_x \phi| \\ &= u(x_0; t_0) \end{aligned} \tag{3.3}$$

Now, $|D_x\phi| = -\frac{|x|}{2t}\phi$ which gives,

$$\begin{aligned} k_\phi &= \int_{\partial U(r)} u|D_x \phi| \\ &= \int_{\partial U(r)} u\frac{\phi}{2t}|x| \\ &= \int_{\partial U(r)} u\frac{|x|}{2t} r \\ \therefore u(x_0; t_0) &= \frac{r}{2}\int_{\partial U(r)} u\frac{|x|}{t} \end{aligned} \tag{3.4}$$

This gives us the surface form of the mean value theorem for the heat equation like we had in the laplace case. Now again I use a ϕ specific method. Multiplying by $\frac{1}{r^2}$ on both sides,

$$\begin{aligned}
\frac{1}{r^2}\int_{\partial U(r)} u\frac{|x|}{2t}r &= -k_\phi\frac{1}{r^2} \\
\therefore \int_{\partial U(r)} u\frac{|x|}{2t}\frac{1}{r} &= -k_\phi\frac{1}{r^2}
\end{aligned}$$

Integrating from R to ∞,

$$\begin{aligned}
\int_R^\infty \int_{\partial U(r)} u\frac{|x|}{2t}\frac{1}{r} &= -k_\phi\int_R^\infty \frac{1}{r^2} \\
&= \frac{k_\phi}{R}
\end{aligned}$$

But from (1.4),

$$\begin{aligned}
\int_R^\infty \int_{\partial U(r)} u\frac{|x|}{2t}\frac{1}{r} &= \int_{U(R)} u\frac{|x|}{2t}\frac{1}{r}|D_x\phi| \\
&= \int_{U(R)} u\frac{|x|}{2t}\frac{1}{r}(r\frac{|x|}{2t}) \\
&= \int_{U(R)} u\frac{|x|^2}{4t^2} \\
\therefore u(x_0;t_0) &= \frac{R}{4}\int_{U(R)} u\frac{|x|^2}{t^2} \qquad (3.5)
\end{aligned}$$

Equations 3.5 and 3.4 are the mean value formulae pair for the Heat Equation, similar to the mean value formulae pair for the Laplace Equation. Both of these and other many mean value formulas are possible from a general mean value formula on the surface of a level set v given by 3.2.

I couldn't prove the limit in 3.3. I got close to it by approximating the integral by taking it on the cylinder enclosing it, but i think it doesn't converge to the given integral in the limit. My approximation gave me

$$k_\phi = \left(n\alpha(n)\left(\frac{n}{2\pi e}\right)^{n/2}\frac{e}{2}\right)u(x_0;t_0)$$