Math 6341, Final Exam: Various Topics Name: ____________________

1. (25 points) (3.5.9-10, Hamilton-Jacobi PDE)

(a) Define the convex dual (Legendre transform) of a function $H : \mathbb{R}^n \to \mathbb{R}$.

(b) Write down the Hopf-Lax formula associated with the IVP

$$\begin{cases} u_t + H(Du) = 0 \\ u(x,0) = u_0(x). \end{cases}$$

(c) Give conditions on the Hamiltonian H and the initial function u_0 under which your formula from part (b) provides a solution of the IVP. (Explain)

Solution:

(a)

$$H^*(v) = \sup_p \{p \cdot v - H(p)\}.$$

(b) The Hopf-Lax formula is

$$u(x,t) = \min_{\xi \in \mathbb{R}^n} \left\{ tL\left(\frac{x-\xi}{t}\right) + u_0(\xi) \right\}$$

where $L = H^*$ is given in (a) above.

(c) In order for the Hopf-Lax formula to be well defined, we need that L is convex satisfying

$$\lim_{|v| \to \infty} \frac{L(v)}{|v|} = +\infty$$

and u_0 Lipschitz continuous.

The conditions on L will hold if H is convex and satisfies

$$\lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty.$$

The Hopf-Lax formula provides a weak solution for the IVP if we assume in addition that either H is uniformly convex or u_0 is semi-concave.

Under either of these assumptions, the solution obtained will be unique.

Name: ____________________

2. (25 points) (Green's Functions for Laplace's PDE, §2.2.4) If $u, v \in C^2(\bar{\Omega})$ satisfy

$$\begin{cases} \Delta u = f \\ u\big|_{\partial\Omega} = u_0, \end{cases}$$

and

$$\begin{cases} \Delta v = 0 \\ v(\xi)\big|_{\xi\in\partial\Omega} = \Phi(\xi - x)\big|_{\xi\in\partial\Omega}, \end{cases}$$

where Φ is the fundamental solution of Laplace's PDE, then find a formula expressing

$$\int_{\xi\in\partial\Omega} \Phi(\xi - x) Du(\xi) \cdot n$$

where n is the outward normal to $\partial\Omega$ in terms of f, u_0, v, and Dv.

Solution: By the divergence theorem

$$\int_\Omega (u\Delta v - v\Delta u) = \int_{\partial\Omega} (uDv \cdot n - vDu \cdot n).$$

Substituting from the boundary value problems, this becomes

$$-\int_\Omega vf = \int_{\partial\Omega} u_0 Dv \cdot n - \int_{\xi\in\partial\Omega} \Phi(\xi - x) Du(\xi) \cdot n.$$

Thus,

$$\int_{\xi\in\partial\Omega} \Phi(\xi - x) Du(\xi) \cdot n = \int_\Omega vf + \int_{\partial\Omega} u_0 Dv \cdot n.$$

Name: ____________________

3. (25 points) (4.7.2) Find a separated variables solution of

$$\begin{cases} \Delta u = 0 \text{ on } \mathbb{R}^2 \\ u(x,0) = 0, \quad u_y(x,0) = \sin x. \end{cases}$$

Explain your reasoning carefully.

Solution: Setting $u = f(x)g(y)$, we obtain away from $f = 0$ or $g = 0$ a separation constant λ such that

$$-\frac{f''}{f} = \frac{g''}{g} = \lambda.$$

Thus, we obtain two ODEs

$$f'' = -\lambda f \quad \text{and} \quad g'' = \lambda g.$$

The first boundary condition gives $f(x)g(0) = 0$ from which we conclude $g(0) = 0$, since $f(x) = 0$ cannot lead to a solution satisfying the other boundary condition.

The second boundary condition is $f(x)g'(0) = \sin x$. Therefore,

$$f(x) = \frac{\sin x}{g'(0)}.$$

It follows from this that $f'' = -f$. In view of the first ODE, we must have $\lambda = 1$. The second ODE with the first boundary condition then yeilds

$$g(y) = g'(0) \sinh y.$$

The solution is thus,

$$u(x,y) = f(x)g(y) = \sinh y \sin x.$$

Name: ____________________

4. (25 points) (Fourier Transform, §4.3.1)

(a) Give an example showing that $L^1(\mathbb{R}^n)$ is not a subset of $L^2(\mathbb{R}^n)$. Justify your assertion.

(b) Give an example showing that $L^2(\mathbb{R}^n)$ is not a subset of $L^1(\mathbb{R}^n)$. Justify your assertion.

(c) For $u \in L^2(\mathbb{R}^n)\backslash L^1(\mathbb{R}^n)$, assume there are two sequences of functions $u_j \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $|u_j - u|_{L^2} \to 0$ and $\tilde{u}_j \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $|\tilde{u}_j - u|_{L^2} \to 0$. Using Plancharel's Theorem, it can be shown that there are functions v and $\tilde{v}$ both in $L^2(\mathbb{R}^n)$ with

$$|\hat{u}_j - v|_{L^2} \to 0 \quad \text{and} \quad |\hat{\tilde{u}}_j - \tilde{v}|_{L^2} \to 0.$$

Show that $v = \tilde{v}$.

Solution:

(a)

$$u(x) = \begin{cases} 1/|x|^{n-1/2}, & 0 < |x| < 1 \\ 0, & x = 0,\ |x| \geq 1 \end{cases}$$

has $u \in L^1\backslash L^2$.

$$\begin{aligned} \int |u| &= \int_0^1 \left(\int_{\partial B_r} \frac{1}{r^{n-1/2}} \right) dr \\ &= n\omega_n \int_0^1 r^{-1/2}\, dr \\ &= 2n\omega_n r^{1/2} \Big|_{r=0}^{1} \\ &= 2n\omega_n \\ &< \infty. \end{aligned}$$

$$\begin{aligned} \int |u|^2 &= \int_0^1 \left(\int_{\partial B_r} \frac{1}{r^{2n-1}} \right) dr \\ &= n\omega_n \int_0^1 r^{-n}\, dr \\ &= n(1-n)\omega_n r^{-n+1} \Big|_{r=0}^{1} \\ &= +\infty. \end{aligned}$$

(b)

$$u(x) = \begin{cases} 1/|x|^n, & |x| > 1 \\ 0, & |x| \leq 1 \end{cases}$$

Name: ____________________

has $u \in L^2 \backslash L^1$.

$$\begin{aligned}\int |u| &= \int_1^\infty \left(\int_{\partial B_r} \frac{1}{r^n} \right) dr \\ &= n\omega_n \int_1^\infty (1/r)\, dr \\ &= n\omega_n \log(r)\Big|_{r=1}^\infty \\ &= \infty.\end{aligned}$$

$$\begin{aligned}\int |u|^2 &= \int_1^\infty \left(\int_{\partial B_r} \frac{1}{r^{2n}} \right) dr \\ &= n\omega_n \int_1^\infty r^{-n-1}\, dr \\ &= -\omega_n r^{-n}\Big|_{r=1}^\infty \\ &= \omega_n \\ &< \infty.\end{aligned}$$

(c)

$$\begin{aligned}|\tilde{v} - v|_{L^2} &= \lim_{j\to\infty} |\hat{\tilde{u}}_j - \hat{u}_j|_{L^2} \\ &= \lim_{j\to\infty} |\widehat{\tilde{u}_j - u_j}|_{L^2} \\ &= \lim_{j\to\infty} |\tilde{u}_j - u_j|_{L^2} \\ &\le \lim_{j\to\infty} |\tilde{u}_j - u|_{L^2} + |u - u_j|_{L^2} \\ &= 0.\end{aligned}$$

The first equality uses the continuity of the L^2 norm which can be further justified by the L^2 triangle inequality as follows

$$||g|_{L^2} - |f|_{L^2}| \le |g - f|_{L^2}.$$

Thus, when g is close to f in L^2, the norms of g and f are also close to each other.

The third equality uses Plancharel's Theorem.