15. $\{f_n\} \subseteq L^+$, $f_n \searrow f$, and $\int f_1 < \infty$, then $\int f = \lim \int f_n$.

pf). let $g_n = f_1 - f_n$.

$\Rightarrow g_n \geq 0$ & $g_n \nearrow f_1 - f$.

By The monotone convergence theorem.

$$\int f_1 - f = \lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \int (f_1 - f_n)$$

$$= \int f_1 - \lim_{n\to\infty} \int f_n.$$

Since $\int f_1 < \infty$,

$$-\int f = -\lim_{n\to\infty} \int f_n \Rightarrow \int f = \lim_{n\to\infty} \int f_n.$$

16. $f \in L^+$, $\int f < \infty$. $\forall \varepsilon > 0$ $\exists$ $E \in M$ s.t. $\mu(E) < \infty$ and

$$\int_E f > \left(\int f\right) - \varepsilon.$$

pf). let $f_n(x) = f(x) \cdot \chi_{E_n}(x)$, $E_n = \{x : f(x) \geq \frac{1}{n}\}$

Then $f_n \nearrow f$ and $f \geq 0$.

By the monotone convergence theorem.

$$\infty > \int f = \lim_{n \to \infty} \int f_n$$

$\Rightarrow$ $\forall \varepsilon > 0$ $\exists$ n_0 s.t. $\int f_{n_0} > \int f - \varepsilon$

$$\therefore \int f_{n_0} = \int f \cdot \chi_{E_{n_0}} = \int_{E_{n_0}} f > \int f - \varepsilon,$$

and

$$\infty > \int f \geq \int_{E_{n_0}} f \geq \frac{1}{n_0} \cdot \mu(E_{n_0}).$$

$\Rightarrow$ $\mu(E_{n_0}) < \infty$. //

Id. f_n ; measurable. $f_n \geq -g$. with $g \in L^+ \cap L^1$.

Then,

$$\int \liminf f_n \leq \liminf \int f_n.$$

pf) $f_n + g \geq 0$.

By Fatou's lemma.

$$\int \liminf (f_n + g) \leq \liminf \int (f_n + g)$$

$$\int \liminf (f_n + g) = \int \liminf f_n + g$$

$$= \int \liminf f_n + \int g. \quad ①$$

and $\liminf \left(\int f_n + g\right) = \liminf \left(\int f_n + \int g\right)$

$$= \left(\liminf \int f_n\right) + \int g \quad ②$$

① + ② $\Rightarrow$ $\int \liminf f_n + \int g \leq \liminf \int f_n + \int g$.

⊕ $\int g < \infty$. $\Rightarrow$ $\int \liminf f_n \leq \liminf \int f_n$ //