12. If $f \in L^+$ and $\int f < \infty$, then $\{x : f(x) = \infty\}$ is a null set and $\{x : f(x) > 0\}$ is a σ-finite.

pf) (1) let $E = \{x : f(x) = \infty\}$.

$$\infty > \int f\, d\mu \geq \int_E f\, d\mu = \infty \cdot \mu(E)$$

$$\therefore \quad \mu(E) = 0.$$

(2) let $E_n = \{x : f \geq \frac{1}{n}\}$ then.

$$\infty > \int f\, d\mu \geq \int_{E_n} f\, d\mu \geq \frac{1}{n} \cdot \mu(E_n) \Rightarrow \mu(E_n) < \infty \quad \forall n.$$

and

$$E = \{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$$

$$\therefore \quad E \text{ is } \sigma\text{-finite}.$$

13. $\{f_n\} \subseteq L^+$, $f_n \to f$ pointwise, $\int f = \lim \int f_n < \infty$. Then $\int_E f = \lim \int_E f_n \quad \forall E \in M$.

However, this need not be true if $\int f = \lim \int f_n = \infty$.

pf). (1) $$\int_E f = \int f \cdot \chi_E \leq \liminf \int f_n \cdot \chi_E = \liminf \int_E f_n \quad - ①$$

By the same reason,

$$\int_{E^c} f \leq \liminf \int_{E^c} f_n$$

But,

$$\int_{E^c} f = \int f - \int_E f, \quad \int_{E^c} f_n = \int f_n - \int_E f_n.$$

$$\therefore \quad \int f - \int_E f \le \liminf \left(\int f_n - \int_E f_n \right)$$

$$= \lim \int f_n + \liminf \left(-\int_E f_n \right)$$

$$= \int f - \limsup \int_E f_n. \quad - ②$$

$$\Leftrightarrow \quad \int_E f \ge \limsup \int_E f_n. \quad \left(\because \int f < \infty \right)$$

$\therefore$ By ①, ②. $$\int_E f = \lim \int_E f_n.$$

(2) If $\int f = \lim \int f_n = \infty$.

Ex).
$$f = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0. \end{cases}$$

$$f_n = \begin{cases} 1 & x \ge 0 \\ \frac{1}{n} & x < 0 \end{cases}$$

$$\Rightarrow \quad \int_{x<0} f = 0, \quad \int_{x<0} f_n = \infty.$$

14. $f \in L^+$. let $\lambda(E) = \int_E f\, d\mu$. for $E \in M$. Then. λ is a measure. and for any $g \in L^+$, $\int g\, d\lambda = \int fg\, d\mu$.

pf). (1). ① $\lambda(\emptyset) = \int_\emptyset f\, d\mu = 0$.

② $\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \int_{\bigcup E_n} f\, d\mu \leq \sum_{n=1}^{\infty} \int_{E_n} f\, d\mu = \sum_{n=1}^{\infty} \lambda(E_n)$.

$\therefore$ λ is a measure.

(2) $\forall$ $g \in L^+$.

Supp $g = \sum_{i=1}^{n} a_i \chi_{E_i}$. $E_i \cap E_j = \emptyset$ if $i \neq j$.

$\Rightarrow$ $\int g\, d\lambda = \int \sum_{i=1}^{n} a_i \chi_{E_i}\, d\lambda = \sum_{i=1}^{n} a_i \lambda(E_i) = \sum_{i=1}^{n} a_i \int_{E_i} f\, d\mu$.

$= \int g \cdot f\, d\mu$. //

for general $g \in L^+$.

we can approximate g by increasing non negative simple ft's g_n, that is,

$0 \leq g_1 \leq g_2 \leq \cdots \leq g_n \leq \cdots \leq g$.

ex) $g_n(x) = \begin{cases} n & x \in \{ g(x) \geq n \}. \\ \frac{k}{2^n} & x \in \{ \frac{k}{2^n} \leq g(x) < \frac{k+1}{2^n} \}. \end{cases}$

Then.

$\int g\, d\lambda \overset{M.C.T}{=} \lim_{n\to\infty} \int g_n\, d\lambda = \lim_{n\to\infty} \int f \cdot g_n\, d\mu \overset{M.C.T}{=} \int fg\, d\mu$.

18. $\{f_n\} \subseteq L^1(\mu)$, and $f_n \rightrightarrows f$.

(a) If $\mu(X) < \infty$; then $f \in L^1(\mu)$ and $\int f_n \to \int f$.

(pf) Given $\varepsilon = 1$, $\exists$ n_0 s.t. $|f_n - f| \le 1$ for all $n \geq n_0$

$$\therefore \int |f| = \int |f - f_n + f_n| \le \int |f - f_n| + |f_n|.$$

$$\le \mu(X) + \int |f_n|$$

$$= \mu(X) + \|f_n\|_{L^1} < \infty.$$

$\therefore$ $f \in L^1$.

and $\forall \varepsilon > 0$. $\exists$ n_0 s.t. $|f_n - f| < \varepsilon$ for all $n > n_0$.

$$\therefore \left| \int f_n - \int f \right| = \left| \int f_n - f \right| \le \int |f_n - f| \le \varepsilon \cdot \mu(X) \quad \forall n > n_0.$$

Since $\varepsilon > 0$ is arbitrary,

$$\int f_n \to \int f \quad //$$

(b) If $\mu(X) = \infty$, the conclusion of (a) can fail.

(ex). $f_n(x) = \begin{cases} \frac{1}{x} & 1 \le x < n \\ 0 & x \geq n. \end{cases}$

$f(x) = \begin{cases} \frac{1}{x} & 1 \le x \\ 0 & x < 1. \end{cases}$

Then trivially f_n converges uniformly to f. but

$$\int f = \int_1^\infty \frac{1}{x}\,dx = \infty. \notin L^1. \quad \text{but} \quad f_n \in L^1 \text{ for all } n.$$