1. $f: X \to \bar{R}$, $Y = f^{-1}(R)$. Then f is measurable iff $f^{-1}(\{-\infty\}) \in M$, $f^{-1}(\{\infty\}) \in M$. and f is measurable on Y.

$(\Rightarrow)$ $f^{-1}(\{-\infty\}) = f^{-1}\left(\bigcap_{n=1}^{\infty} [-\infty, -n)\right) = \bigcap_{n=1}^{\infty} f^{-1}[-\infty, -n)$; measurable.

$f^{-1}(\{\infty\})$: same argument.

let $f|_Y = g$. $\Rightarrow$ $g: Y \to R$. then $\forall a \in R$.

$$g^{-1}((a,\infty)) = f^{-1}((a,\infty]) \setminus f^{-1}(\{\infty\}) \text{ : measurable.}$$

$\therefore$ f is measurable on Y.

$(\Leftarrow)$. $\forall a \in R$.

$$f^{-1}((a,\infty]) = f|_Y^{-1}((a,\infty)) \cup f^{-1}(\{\infty\}) \text{ : measurable.}$$

$\therefore$ f is measurable.

2. $f, g: X \to \bar{R}$ are measurable.

(a) fg is measurable.

$$fg^{-1}(\{-\infty\}) = [f^{-1}(\{\infty\}) \cap g^{-1}([-\infty, 0))] \cup [f^{-1}(\{-\infty\}) \cup g^{-1}(0,\infty]]$$
$$\cup [f^{-1}[-\infty, 0) \cap g^{-1}(\{\infty\})] \cup [f^{-1}(0,\infty] \cap g^{-1}\{-\infty\}].$$

; measurable.

$fg^{-1}(\{\infty\})$: measurable (a ...)

let $f^{-1}(\mathbb{R}) = A$, $g^{-1}(\mathbb{R}) = B$.

$\Rightarrow$ fg is measurable on $A \cap B$. (proposition 2.6)

and $(fg)^{-1}(\mathbb{R}) = [A \cap B] \cup \underbrace{[f^{-1}(\{-\infty, \infty\}) \cap g^{-1}(\{0\})]}_{=C}$

$\cup \underbrace{[g^{-1}(\{\infty, \infty\}) \cap f^{-1}(\{0\})]}_{=D}$

$$fg|_{(fg)^{-1}(\mathbb{R})} = fg|_{A \cap B} + 0 \cdot \chi_{C \cup D} \quad ; \text{ measurable.}$$

$\therefore$ By the previous exercise, fg is measurable.

b) Fix $a \in \overline{\mathbb{R}}$. define $h(x) = \begin{cases} a & f(x) = -g(x) = \pm\infty. \\ f(x) + g(x) & \text{otherwise.} \end{cases}$

$\Rightarrow$ h is measurable.

Sol). let $C = \{x : f(x) = -g(x) = \pm\infty\}$; measurable.

$h^{-1}(\{\infty\}) = [f^{-1}(\{\infty\}) \cap g^{-1}((-\infty, \infty]))] \cup [f^{-1}((-\infty, \infty]) \cap g^{-1}(\{\infty\})]$

; measurable.

$h^{-1}(\{-\infty\})$; measurable also.

$\forall$ $b \in \mathbb{R}$.

$h^{-1}((b, \infty]) = h^{-1}((b, \infty)) \cup h^{-1}(\{\infty\})$

$$= \begin{cases} h^{-1}(\{\infty\}) \cup (f+g)|_{A\cap B}^{-1}(b,\infty) & \text{if } a \le b. \\ h^{-1}(\{\infty\}) \cup (f+g)|_{A\cap B}^{-1}(b,\infty) \cup C & \text{if } b < a. \end{cases}$$

measurable.

$\Rightarrow$ $\therefore$ h is measurable.

3. f_n ; sequence of measurable funtions on X, then $\{x : \lim f_n(x) \text{ exists}\}$ is a measurable set.

Sol).

For a fixed $\varepsilon > 0$. Define.

$$A_n^{\varepsilon} = \{x : |f_i(x) - f_j(x)| \le \varepsilon \ \text{, for all } i, j \ge n\}.$$

$$= \bigcap_{i,j\ge n} \{|f_i - f_j| \le \varepsilon\} \quad ; \text{measurable.}$$

Then

$$A := \{x : \lim_{n\to\infty} f_n(x) = a \ \text{ for some } a \in \mathbb{R}\}$$

$$= \bigcap_{k=1}^{\infty} \liminf_{n\to\infty} A_n^{\frac{1}{k}} \quad ; \text{measurable.}$$

$$B := \{x : \lim_{n\to\infty} f_n(x) = \infty\} = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \{x : f_n(x) > k \ \text{, for all } n \ge m\}$$

$C := \{x : \lim_{n\to\infty} f_n(x) = -\infty\} = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \{x : f_n(x) \leq -k, \text{ for all } n \geq m\}$

$\therefore \{x : \lim f_n(x) \text{ exists}\} = A \cup B \cup C.$

4. If $f : X \to \overline{\mathbb{R}}$ and $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

pf) It is enough to show that $f^{-1}((a,\infty]) \in \mathcal{M}$ for each $a \in \mathbb{R}$.
For fixed $a \in \mathbb{R}$, $\exists$ $r_n \in \mathbb{Q}$ such that $a < \cdots < r_{n+1} < r_n < \cdots < r_1$ and $\lim_{n\to\infty} r_n = a$.

$\Rightarrow f^{-1}((a,\infty]) = f^{-1}\left(\bigcap_{n=1}^{\infty}(r_n, \infty]\right) = \bigcap_{n=1}^{\infty} f^{-1}((r_n,\infty]) \in \mathcal{M}.$

8. $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

(pf) $\forall$ $a \in \mathbb{R}$. $f^{-1}((a,\infty)) = (b, \infty)$. or $[c, \infty)$.
for some b, c ; measurable.

10. The following implications are valid iff the measure μ is complete.

(a) If f is measurable and $f=g$ μ-a.e, then g is measurable.

(b) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e, then f is measurable.

($\Rightarrow$) If μ is not complete, then there exists a measurable set N. such that $\mu(N)=0$ with $A \subset N$, A is not measurable.

Then $\chi_N = \chi_A$ μ-a.e ; and χ_N is measurable, but χ_A is not measurable.

$\therefore$ (a) implies that μ is complete.

Take $f_n = \chi_N$, then $f_n = \chi_N \longrightarrow \chi_A$ μ a.e.

& f_n is measurable but $f = \chi_A$ is not measurable.

$\therefore$ (b) implies that μ is complete.

($\Leftarrow$) (a) $\forall$ $r \in \mathbb{R}$.

$$g^{-1}((a,\infty]) = \left(f^{-1}(a,\infty]\right) \triangle N \quad \text{for some } N \text{ s.t. } \mu(N)=0$$

$\Rightarrow$ measurable $\because$ f is measurable & N is also

$\therefore$ μ is complete implies that if $f=g$ μ-a.e & f is measurable $\Rightarrow$ g is measurable.

(b). let $A = \{x : f_n(x) \to f(x)\}$, then $A^c \subset N$ for some $\underset{\sim}{N}$.

$\therefore$ N is measurable and $\mu(N)=0$.

$\Rightarrow$ $f_n \to f$ on $X \setminus N$. let $X \setminus N = B$.

$\Rightarrow$ $\chi_B \cdot f_n(x) \to \chi_B f(x)$ $\forall x \in X$.

& $\chi_B \cdot f_n$ is measurable for all n.

$\Rightarrow$ $\chi_B \cdot f$ is measurable by proposition 2.7.

& $\chi_B f = f$ μ-a.e.

$\therefore$ f is measurable by (a).

$\therefore$ μ is complete implies that if f_n is measurable for $n \in \mathbb{N}$. if $f_n \to f$ μ-a.e then f is measurable.