27. C : Cantor set.

$\Rightarrow$ C is compact, nowhere dense and totally disconnected.
& C has no isolated points.

pf) ① C is countable intersection of closed sets $\Rightarrow$ closed.
& bounded

$\therefore$ C is compact ($\because$ closed & bounded subset of $\mathbb{R}^1$).

② $\forall$ $x, y \in C$. with $x < y$.

$\exists$ n $\ni$. $\frac{1}{3^{n+1}} < y - x$ $\Rightarrow$ $x < \frac{j}{3^{n+1}} < \frac{j+1}{3^{n+1}} < y$.
for some j

$\Rightarrow$ $\left(\frac{j}{3^{n+1}}, \frac{j+1}{3^{n+1}}\right) \cap C = \emptyset$.

$\therefore$ $\exists$ $z \in \left(\frac{j}{3^{n+1}}, \frac{j+1}{3^{n+1}}\right)$ & $x < z < y$. $z \notin C$.

This shows C is totally disconnected

and also C can not contain any open interval

$\Rightarrow$ interior of $C = \emptyset$.

$\Rightarrow$ C is nowhere dense

③ let $x \in C$. Then there are two cases.

(i) $x = \sum_{j=1}^{\infty} a_j \frac{1}{3^j}$ with $a_j \neq 1$ for all j and

$a_j = 2$ for infinitely many j's

Define $x_n = \sum_{j=1}^{n} a_j \frac{1}{3^j} \in C$. and $x_n \to x$

(ii) $x = \sum_{j=1}^{k} a_j \frac{1}{3^j}$ for some k and $a_k = 2$.

Define $x_n = \sum_{j=1}^{k} a_j \frac{1}{3^j} + \frac{2}{3^n}$ for $n > k$.

Then $x_n \in C$ and $x_n \to x$.

$\therefore$ C has no isolated points.

28. F : increasing and right continuous. μ_F : associated measure.

Then.

$$\mu_F(\{a\}) = \mu_F\left(\bigcap_{n=1}^{\infty} (a-\tfrac{1}{n}, a]\right) = \lim_{n\to\infty} \mu_F\left((a-\tfrac{1}{n}, a]\right)$$

$$= \lim_{n\to\infty} F(a) - F(a-\tfrac{1}{n})$$

$$= F(a) - F(a^-).$$

$$\mu_F([a,b)) = \mu_F(\{a\} \cup (a,b)) = \mu_F(\{a\}) + \mu_F((a,b))$$

$$= \mu_F(\{a\}) + \mu_F((a,b] \setminus \{b\})$$

$$= \mu_F(\{a\}) + \mu_F((a,b]) - \mu_F(\{b\})$$

$$= F(a) - F(a^-) + F(b) - F(a) - F(b) + F(b^-)$$

$$\mu_F([a,b]) = \mu_F(\{a\} \cup (a,b]) = \mu_F(\{a\}) + \mu_F((a,b])$$
$$= F(a) - F(a^-) + F(b) - F(a)$$
$$= F(b) - F(a^-).$$

$$\mu_F((a,b)) = \mu_F((a,b] \setminus \{b\}) = \mu_F((a,b]) - \mu_F(\{b\})$$
$$= F(b) - F(a) - F(b) + F(b^-)$$
$$= F(b^-) - F(a).$$

29. E : Lebesgue measurable set.

a. If $E \subset N$, N in §1.1, then $m(E) = 0$.

pf) let $R = \mathbb{Q} \cap [0,1)$. and for each $r \in R$.

Define,

$$E_r = \{x+r \,;\, x \in E \cap [0,1-r)\} \cup \{x+r-1 \,;\, x \in E \cap [1-r,1)\}.$$

$\Rightarrow$ E_r ; measurable subset of $[0,1)$ for all $r \in R$.

and $E_r \cap E_s = \emptyset$ if $r \neq s$.

moreover, $m(E) = m(E_r)$ for all $r \in R$. Since m is translation invariant.

let $F = \bigcup_{r \in R} E_r$ $\Rightarrow$ $F \subseteq [0,1)$ & measurable.

with $m(F) = m\left(\bigcup_{r \in R} E_r\right) = \sum_{r \in R} m(E_r) = \sum_{r \in R} m(E) \leq 1$

$\therefore\ m(E) = 0$ (otherwise $m(F) = \infty < 1$ ※).

b. If $m(E) > 0$, then E contains a non measurable set.

pf). Without loss of generality, we can assume $E \subset [0,1]$.

$\Rightarrow\ E = \bigcup_{r \in R} E \cap N_r$.

if $E \cap N_r$ is measurable for all $r \in R$, then.

$$m(E) = m\Big[\bigcup_{r\in R} (E \cap N_r)\Big] = \sum_{r \in R} m(E \cap N_r) \underset{\uparrow}{=} \sum_{r\in R} 0 = 0.$$

(From a. $m(E \cap N_r) = 0$.)

but $m(E) > 0$. (※).

$\therefore$ at least one of $E \cap N_r$ should be non-measurable //.

Problem 1.5.

1. μ ; locally finite Borel measure on $\mathbb{R}$.

$$g(x) = \begin{cases} -\mu(x, 0] & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \mu(0, x] & \text{if } x > 0. \end{cases}$$

g is non decreasing ; Trivial from the definition of g.

$\inf_{x > a} g(x) = g(a)$; i. $\underset{\sim\sim}{a > 0}$

$g(a) = \mu((0,a]) = \mu(\bigcap^{\infty} (0, a + \frac{1}{n}]))$

$$= \lim_{n \to 0} g(a + \tfrac{1}{n}) \quad - ①$$

$\therefore \quad g(a) = \inf_{x > a} g(x) \quad$ (① + g is nondecreasing).

2. $a = 0$.

3. $a < 0$

$\Rightarrow$ Can be proved similarly.

2. a.

$$\mu_0 E = \sum_{j=1}^{k} [g(b_j) - \lim_{x \downarrow a_j} g(x)]$$

$$= \sum_{j=1}^{k} [g(b_j) - g(a_j)] \quad \Leftarrow ☺ \quad g \text{ is right-continuous.}$$

if $E = \cup (a_j, b_j]$

$$\mu_0 E = \sum_{j=1}^{k-1} [g(b_j) - \lim_{x \downarrow a_j} g(x)] + \lim_{x \uparrow \infty} (g(x) - g(a_k))$$

$$= \sum_{j=1}^{k-1} [g(b_j) - g(a_j)] + \lim_{x \uparrow \infty} (g(x) - g(a_k)).$$

Then μ_0 is a premeasure ; Almost same as Lebesgue measure
$\Rightarrow$ Easy.

b. There is a unique (locally finite) Borel measure μ_g on $\mathbb{R}$ such that $\mu_g(a,b] = g(b) - g(a)$.

pf) Next page.

pf) let ν_g be another Borel measure on $\mathbb{R}$ such that

$$\nu_g((a,b]) = g(b) - g(a).$$

$\Rightarrow$ I want to show $\mu_g(B) = \nu_g(B)$ for all Borel sets B.

Since Borel Algebra is generated by open sets, it is enough to show

$$\mu_g(O) = \nu_g(O) \quad \text{for all open sets } O.$$

and every open set $O = \bigcup_{j=1}^{\infty} I_j$, I_j : open interval.

in $\mathbb{R}$

$\Rightarrow$ It is enough to show that

$$\mu_g(I) = \nu_g(I) \quad \text{for all open intervals } I.$$

(1) $I = (a,b)$.

$$\mu_g((a,b)) = \mu_g((a,b] \setminus \{b\}) = \mu_g((a,b]) - \mu_g(\{b\})$$

$$= g(b) - g(a) - \mu_g\left(\bigcap_{n=1}^{\infty} (b-\tfrac{1}{n}, b]\right)$$

$$= g(b) - g(a) - \lim_{n\to\infty} \mu_g((b-\tfrac{1}{n}, b])$$

$$= g(b) - g(a) - \lim_{n\to\infty} [g(b) - g(b-\tfrac{1}{n})]$$

$$= \nu_g((a,b]) - \lim_{n\to\infty} \nu_g((b-\tfrac{1}{n}, b])$$

$$= \nu_g((a,b]) - \nu_g\left(\bigcap^{\infty} (b-\tfrac{1}{n}, b]\right)$$

$= \nu_g((a,b]) - \nu_g(\{b\}) = \nu_g\ (a,b).$

(2). $I = (a, \infty)$.

$$\mu_g((a,\infty)) = \mu_g\left(\bigcup_{n=1}^{\infty} (a, a+n]\right)$$
$$= \lim_{n\to\infty} \mu_g((a, a+n])$$
$$= \lim_{n\to\infty} g(a+n) - g(a)$$
$$= \lim_{n\to\infty} \nu_g((a, a+n])$$
$$= \nu_g\ (a, \infty)$$

From (1), (2). $\mu_g = \nu_g$. as a Borel measure.

3. Lebesgue - Stieltjes measures are Borel regular.

pf). Refer Theorem 1.18.