20. μ^* : outer measure on X.

M^* : σ-algebra of μ^*-measurable sets.

$\bar{\mu} = \mu^*|_{M^*}$

μ^+ : outer measure induced by $\bar{\mu}$.

(a). $E \subseteq X$, we have $\mu^*(E) \le \mu^+(E)$. with equality iff $\exists\ A \in M^*$ with $A \supset E$ and $\mu^*(A) = \mu^*(E)$.

pf) $\mu^+(E) = \inf \{ \sum_{i=1}^{\infty} \bar{\mu}(A_i) : E \subseteq \bigcup_{i=1}^{\infty} A_i, \ A_i \in M^* \}$

& $\mu^*(E) \le \mu^*(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mu^*(A_i) = \sum_{i=1}^{\infty} \bar{\mu}(A_i)$

$\therefore$ $\underline{\mu^+(E) \geq \mu^*(E)}$.

and $\forall\ n$, $\exists\ \{A_i^n\} \subseteq M^*$ s.t. $E \subseteq \bigcup_{i=1}^{\infty} A_i^n$.

$$\mu^+(E) \geq \sum_{i=1}^{\infty} \bar{\mu}(A_i^n) - \frac{1}{n}.$$

let $B_n = \bigcup_{i=1}^{\infty} A_i^n$

$\Rightarrow$ $B_n \supseteq E$, $B_n \in M^*$ & $\mu^*(B_n) = \bar{\mu}(B_n) \le \mu^+(E) + \frac{1}{n}$

let $B = \bigcap_{n=1}^{\infty} B_n$.

$\Rightarrow$ $B \supseteq E$, $B \in M^+$ & $\mu^*(B) = \bar{\mu}(B) \le \mu^+(E) \le \mu^+(B)$

$\therefore$ $\mu^*(E) \le \mu^*(B) = \bar{\mu}(B) \le \mu^+(E) \le \mu^+(B) = \bar{\mu}(B)$

$\therefore$ $\mu^+(E) = \mu^*(E)$ iff $\mu^*(B) = \mu^*(E)$, $E \subseteq B$, $B \in M^*$

(b) If μ^* is induced from a premeasure, then $\mu^* = \mu^+$.

pf). For fixed $E \subseteq X$, and n, $\exists$ $A_n \in \mathcal{A}_\sigma$ with $E \subset A_n$ and

$\mu^*(A_n) \le \mu^*(E) + \frac{1}{n}$.

(By 18.(a))

let $A = \bigcap_{n=1}^{\infty} A_n$

$\Rightarrow$ $E \subseteq A \in \mathcal{A}_{\sigma\delta} \subseteq \mathcal{M}^*$. with $\mu^*(E) \le \mu^*(A) \le \mu^*(E)$

$\therefore$ $\mu^*(A) = \mu^*(E)$ with $A \in \mathcal{M}^*$.

by (a). $\mu^*(E) = \mu^+(E)$. //

(c) $X = \{0, 1\}$.

Sol). Define $\mu^* : P(X) \to [0, \infty]$.

$\mu^*(\emptyset) = 0$.

$\mu^*(\{1\}) = 2$

$\mu^*(\{0\}) = 2$

$\mu^*(\{0,1\}) = 3$.

$\Rightarrow$ μ^* : outer measure.

and $\{\emptyset, \{0,1\}\}$: σ-algebra of μ^*-measurable sets.

$\Rightarrow$ μ^+ : outer measure induced by $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$ is following.

$\mu^+(\emptyset) = 0$ $\quad$ $\mu^+(\{0\}) = 3$

$\mu^+(\{1\}) = 3$ $\quad$ $\mu^+(\{0,1\}) = 3$.

$\Rightarrow$ $\mu^* \neq \mu^+$.

30. $E \in \mathcal{L}$ and $m(E) > 0$, for any $\alpha < 1$, $\exists$ an open interval I $\ni$. $m(E \cap I) > \alpha\, m(I)$.

Sol). Without big loss of generality. we can assume

$$E \subseteq (0,1).$$

$$m(E) = \inf \{ \sum_{i=1}^{\infty} m(I_i) : E \subset \bigcup_{i=1}^{\infty} I_i, \quad I_i : \text{disjoint intervals in } (0,1) \}$$

$\Rightarrow$ $\forall\, \varepsilon > 0$, $\exists$ $\{I_i\}$ $\ni$. $E \subseteq \bigcup_{i=1}^{\infty} I_i$, with $m(E) \geq \sum_{i=1}^{\infty} m(I_i) - \varepsilon$

$$\therefore \sum_{i=1}^{\infty} m(I_i) - \varepsilon \leq m(E) = \sum_{i=1}^{\infty} m(I_i \cap E) \quad - ①$$

if for any $\alpha < 1$, all intervals satisfy $m(E \cap I) \leq \alpha\, m(I)$ $-$ ②

Then from ①.

$$\sum_{i=1}^{\infty} m(E_i) - \varepsilon \leq \sum_{i=1}^{\infty} m(I_i \cap E) \leq \alpha \cdot \sum_{i=1}^{\infty} m(I_i) = 0. \quad (\because \forall\, \alpha < 1).$$

$$\Rightarrow \sum_{i=1}^{\infty} m(E_i) \leq \varepsilon \quad \text{※} \quad (\because \varepsilon \text{ is small number})$$

$\therefore$ ② is false.

$\therefore$ for any $\alpha < 1$, $\exists$ I $\ni$. $m(E \cap I) > \alpha\, m(I)$

32. $\{\alpha_j\} \subseteq (0,1)$.

(a). $\prod_1^\infty (1-\alpha_j) > 0$ iff $\sum_1^\infty \alpha_j < \infty$.

pf) Define $f(x) = x + \log(1-x) \quad 0<x<1$

(i) $f(0) = 0$

(ii) $f'(x) = 1 - \frac{1}{1-x} = \frac{-x}{1-x} < 0 \Rightarrow f$ is decreasing.

$\therefore \quad f(x) = x + \log(1-x) \le 0 \quad x \in (0,1)$.

$\therefore \quad \underline{\log(1-\alpha_j) \le -\alpha_j \quad \forall \alpha_j \in (0,1)}$.

$\therefore \prod_1^\infty (1-\alpha_j) > 0$ iff $-\infty < \log\left[\prod_1^\infty (1-\alpha_j)\right] = \sum_{j=1}^\infty \log(1-\alpha_j)$

($\Rightarrow$) $\prod_1^\infty (1-\alpha_j) > 0$ implies $-\infty < \sum_{j=1}^\infty \log(1-\alpha_j) \le -\sum_{j=1}^\infty \alpha_j$

$\therefore \sum_{j=1}^\infty \alpha_j < \infty$.

($\Leftarrow$) $\infty > \sum_{j=1}^\infty \alpha_j$ implies $\log\left[\prod_1^\infty (1-\alpha_j)\right] > -\infty$.

$x \in (0,1)$

(1) $\log(1-x) = \log(1-0) + \frac{-1}{1-c} \cdot x = -\frac{x}{1-c} \quad 0 \le c < x$

↳ Taylor's thm

and since $\sum \alpha_j < \infty$, $\alpha_j \to 0$ as $j \to \infty$.

$\therefore \quad \log(1-\alpha_j) = -\frac{\alpha_j}{1-c_j} \quad 0 < c_j < \alpha_j \to 0$ as $j \to \infty$.

$\therefore \quad \log(1-\alpha_j) \approx k\cdot\alpha_j$ for some fixed k for all j.

$\therefore \infty > \sum \alpha_j$ iff $-\sum \log(1-\alpha_j) < \infty$

iff $\prod_1^{\infty} (1-\alpha_j) > 0$. //

(b). Given $\beta \in (0,1)$ choose $\{\alpha_j\}$ $\ni$. $\prod_1^{\infty}(1-\alpha_j) = \beta$.

Sol). We know that $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

let $a = \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{a\cdot n^2} = 1$.

$$\Leftrightarrow \sum_{n=1}^{\infty} \frac{\log\beta}{a\cdot n^2} = \log\beta.$$

$$\Leftrightarrow \prod_{n=1}^{\infty} \exp\left(\frac{\log\beta}{a n^2}\right) = \beta.$$

$\exp\left(\frac{\log\beta}{an^2}\right) = 1-\alpha_n$; Definition of α_n.

$\Rightarrow \quad 1-\alpha_n = \exp(\;) > 0 \quad \therefore \quad \alpha_n < 1$

and $\frac{\log\beta}{an^2} < 0 \quad (\because 0<\beta<1, \quad an^2>0)$

$\Rightarrow \quad \exp\left(\frac{\log\beta}{an^2}\right) < 1 \Rightarrow 1-\alpha_n < 1 \Rightarrow \alpha_n > 0$.

//.

15. (a). (b). (c). : Easy (even though a little bit time consuming).

(d) : You can use the following property

In 1-dimensional space ($\mathbb{R}^1$), Every open set is a countable disjoint union of open intervals.

i.e. $U = \bigcup_{n=1}^{\infty} I_n$. I_n : open intervals

$I_n \cap I_m = \emptyset$ if $n \neq m$

for any open set $U \subseteq \mathbb{R}^1$.

and a "typo"

$$M^* = \{A \subseteq \mathbb{R} : \mu^* A = \underset{K \subseteq A,\ \text{compact}}{\overset{\inf X}{\sup}}\ mK\}$$

(e) : Easy.

2. $\{0,1\}^{\mathbb{N}}$ is uncountable.

Sol). $\mathcal{C} = \{(a_n)_{n=1}^{\infty} : a_n = 0 \text{ or } 1\}$.

Suppose $\mathcal{C}$ is countable, then $\mathcal{C} = \{\alpha_i : i = 1, \dots, n, \dots\}$.

with $\alpha_i = (a_n^i)_{n=1}^{\infty}$ s.t. $a_n^i = 0$ or 1

$\alpha_1 = a_1^1 \ a_2^1 \cdots \ a_n^1, \cdots$

$\alpha_2 = a_1^2 \ a_2^2 \cdots \ a_n^2 \cdots$

$\vdots$

$\alpha_j = a_1^j \ a_2^j \cdots \ a_n^j \cdots$

$\vdots$

let $\beta = (b_j)_{j=1}^{\infty}$ $\ni$.

$b_1 \neq a_1^1$

$b_2 \neq a_2^2$

$\vdots$

$b_n \neq a_n^n$

$\vdots$

Then $\beta = (b_j)_{j=1}^{\infty} \in \ell$ but $\beta \neq \alpha_i$ for all i.

$\times$

$\therefore$ ℓ is uncountable $\Rightarrow$ $\{0,1\}^{\mathbb{N}}$ is uncountable.