17. μ^* - outer measure. $\{A_j\}$: disjoint μ^* - measurable sets.

$$\Rightarrow \quad \mu^*\left(E \cap \left(\bigcup_1^\infty A_j\right)\right) = \sum_1^\infty \mu^*(E \cap A_j). \quad \forall\ E \subset X.$$

(pf). ① $\mu^*(E \cap \bigcup_1^\infty A_j) = \mu^*\left(\bigcup_1^\infty (E \cap A_j)\right) \leq \sum_1^\infty \mu^*(E \cap A_j).$ — ①

② For any n,

$$\mu^*(E \cap \bigcup_1^\infty A_j) \geq \mu^*(E \cap \bigcup_1^n A_j)$$

$$= \mu^*\left(\bigcup_1^n (E \cap A_j)\right)$$

$\downarrow$ A_1 : μ^* - measurable.

$$= \mu^*\left[A_1 \cap \left(\bigcup_1^n (E \cap A_j)\right)\right] + \mu^*\left[A_1^c \cap \left(\bigcup_1^n (E \cap A_j)\right)\right]$$

A_j's are disjoint $\downarrow$

$$= \mu^*(E \cap A_1) + \mu^*\left[\bigcup_2^n (E \cap A_j)\right]$$

$$\vdots$$

$$= \sum_{j=1}^n \mu^*(E \cap A_j).$$

Since this is true for any fixed n,

$$\mu^*(E \cap \bigcup_1^\infty A_j) \geq \sum_{j=1}^\infty \mu^*(E \cap A_j). \quad — ②$$

From ①, ②

$$\mu^*(E \cap \bigcup_1^\infty A_j) = \sum_{j=1}^\infty \mu^*(E \cap A_j).$$

18. (a). $\mu^*(E) = \inf \{ \sum_{j=1}^{\infty} \mu(A_j) : E \subset \bigcup_{j=1}^{\infty} A_j \}$.

$\therefore$ For fixed $\varepsilon > 0$, $\exists$ $\{A_j\}_1^{\infty}$ $\ni$. $\mu^*(E) \geq \sum_{j=1}^{\infty} \mu(A_j) - \varepsilon$

let $A = \bigcup_{j=1}^{\infty} A_j$, then $A \in \mathcal{A}_\sigma$ and

$$\mu^*(E) + \varepsilon \geq \sum_{j=1}^{\infty} \mu(A_j)$$

$$= \sum_{j=1}^{\infty} \mu^*(A_j) \quad \text{By proposition 1.13}$$

$$\geq \mu^*(A).$$

(b). $\mu^*(E) < \infty$, E is μ^*-measurable iff $\exists$ $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.

(pf) ($\Leftarrow$). $B = E \cup (B \setminus E)$ and

B is μ^*-measurable. ($\because$ $B \in \mathcal{A}_{\sigma\delta}$)

$B \setminus E$ is μ^*-measurable ($\because$ $\mu^*(B \setminus E) = 0$ + Completeness of μ^*)

$\therefore$ $E = B \setminus (B \setminus E)$ is μ^*-measurable.

($\Rightarrow$) $\forall n$, $\exists$ $\{A_i^n\}_{i=1}^{\infty}$ $\ni$. $E \subset \bigcup_{i=1}^{\infty} A_i^n$, $A_i^n \in \mathcal{A}$.

with $\infty > \mu^*(E) \geq \sum_{i=1}^{\infty} \mu(A_i^n) - \frac{1}{n}$.

let $B_n = \bigcup_{i=1}^{\infty} A_i^n$, $B = \bigcap_{n=1}^{\infty} B_n$. $\Rightarrow$ $B \in \mathcal{A}_{\sigma\delta}$, $B \supset E$.

B is μ^*-measurable. By proposition 1.13

$$\mu^*(B) = \mu^*\left(\bigcap_{n=1}^{\infty} B_n\right) \le \liminf_{n\to\infty} \mu^*(B_n) \le \liminf_{n\to\infty} \sum_{i=1}^{\infty} \mu(A_i^n)$$

(Here $\infty > \mu^*(E)$ is used)

$$\le \liminf_{n\to\infty} \mu^*(E) + \frac{1}{n}$$

$$= \mu^*(E).$$

Since B and E are μ^* measurable, $B \setminus E$ is also μ^* measurable.

and $\therefore \mu^*(B\setminus E) = \mu^*(B) - \mu^*(E) = 0$.

(again, this is possible because $\mu^*(E) < \infty$)

(c). If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superflous.

(pf). $X = \bigcup_{i=1}^{\infty} X_i \ni X_i \cap X_j = \phi$ if $i \ne j$ and

$\mu_0(X_i) < \infty$ for all i.

$\Rightarrow E = \bigcup_{i=1}^{\infty} E \cap X_i$ with $\mu^*(E \cap X_i) < \infty$.

$\therefore E$ is μ^*-measurable iff $E \cap X_i$ is μ^*-measurable for all i.

iff $\exists B_i \in A_{\sigma\delta} \ni E_i \subseteq B_i$, $\mu^*(B_i \setminus E_i) = 0$.

iff $E \subseteq B = \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty}\left(\bigcap_{j=1}^{\infty}\bigcup_{k=1}^{\infty} A_{jk}^i\right)$

$$\subseteq \bigcap_{j=1}^{\infty}\left(\bigcup_{i=1}^{\infty}\bigcup_{k=1}^{\infty} A_{jk}^i\right) \in A_{\sigma\delta}$$

with $\mu^*(B\setminus E) \le \mu^*\left(\bigcup_{i=1}^{\infty}(B_i\setminus E_i)\right) \le \sum_{i=1}^{\infty}\mu^*(B_i\setminus E_i) = 0.$ //

19. μ^* : outer measure, μ_0 : premeasure. (finite).

μ_* : inner measure, $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$.

E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$.

(pf) E is μ^*-measurable iff $\exists\ B \in A_{\sigma\delta}$ s.t. $E \subset B$ with

$\mu^*(B\setminus E) = 0$.

$\Rightarrow$

$$\begin{aligned}\mu_*(E) &= \mu_0(X) - \mu^*(E^c)\\ &= \mu_0(X) - (\mu^*((E^c\cap B)\cup B^c))\\ &\ge \mu_0(X) - \mu^*(B^c) - \underbrace{\mu^*(E^c\cap B)}_{=0}\\ &= \mu^*(B) \qquad \left(\text{Since } B \text{ is } \mu^*\text{-measurable and } \mu_0(X) = \mu^*(X)\right).\\ &\ge \mu^*(E). \quad — ①\end{aligned}$$

and.

$$\mu_0(X) = \mu^*(X) \le \mu^*(E) + \mu^*(E^c)$$

$$\therefore\ \mu^*(E) \ge \mu^*(X) - \mu^*(E^c) = \mu_0(X) - \mu^*(E^c)$$
$$= \mu_*(E) \quad — ②$$

from ①, ② $\quad \mu_*(E) = \mu^*(E)$.

2. Assume that

$$\mu_* A = \sup_{A \supset E \in \mathcal{A}} \mu E . \qquad \mu \text{ : measure on } \mathcal{A}.$$

Find a set $E_* \in A$ $\ni$. $E_* \subset A$ and $\mu E_* = \mu_* A$. for given A.

Sol). $\forall n$, $\exists$ E_n $\ni$.

$$\mu(E_n) \geq \mu_*(A) - \frac{1}{n} \quad , \quad E_n \in \mathcal{A}. \quad E_n \subseteq A.$$

let $E_* = \bigcup_{n=1}^{\infty} E_n$., $\quad E_* \subseteq A$.

$$\Rightarrow \quad \mu_*(A) \geq \mu(E_*) \geq \limsup_{n\to\infty} \mu(E_n) \geq \limsup_{n\to\infty} \left(\mu_*(A) - \frac{1}{n}\right) = \mu_*(A).$$

$$\therefore \quad \mu(E_*) = \mu_* A . \quad //.$$

The Caratheodory condition is equivalent to $\mu_* A = \mu^* A$.

Sol). From above, $\exists$ $E_* \subseteq A$ $\ni$. $E_* \in \mathcal{A}$ and $\mu(E_*) = \mu_*(A)$

Since μ is a measure, $\exists$ $E^* \supseteq A$ $\ni$. $E^* \in \mathcal{A}$ and

$$\mu(E^*) = \mu^*(A).$$

ie $\quad E_* \subseteq A \subseteq E^*$

with. $\mu^*, \mu_* \in \mathcal{A}$.

$\mu_* A = \mu^* A$ iff $\mu(E_*) = \mu(E^*)$, for some E_*, E^* : measurable.

$\ni$. $E_* \subseteq A \subseteq E^*$

$\Rightarrow$ For fixed $B \subseteq X$,

$$\mu^*(B \cap A) \le \mu^*(B \cap E^*) \le \mu(G \cap E^*) \qquad \forall\ B \subseteq G, \quad G \in \mathcal{A}.$$

$$\mu^*(B \cap A^c) \le \mu^*(B \cap (E_*)^c) \le \mu(G \cap (E_*)^c)$$

and

$$\mu(G) = \mu(G \cap E^*) + \mu(G \cap (E^*)^c)$$

$$= \mu(G \cap E^*) + \mu(G \cap (E_*)^c) \quad (\because\ \mu(E_*) = \mu(E^*)).$$

$$\ge \mu^*(B \cap A) + \mu^*(B \cap A^c) \quad -\ ①.$$

① is true for all $G \in \mathcal{A}$ $\ni$ $B \subseteq G$.

$$\therefore \quad \mu^*(B) = \inf_{B \subseteq G \in \mathcal{A}} \mu(G) \ge \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

$$\therefore \quad \mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

$\therefore$ $\mu_* A = \mu^* A$ implies $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$ $\forall\ B \subseteq X$.

<u>Next,</u>

If $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$ for all $B \subseteq X$, then.

$$\mu^*(X) = \mu^*(A) + \mu^*(A^c) \text{ is also true.} \quad -\ ①$$

As we saw before, $\exists$ $C, D \in \mathcal{A}$

$\ni$ $A \subseteq C$ and $\mu^*(A) = \mu(C)$

$A^c \subseteq D$ $\qquad \mu^*(A^c) = \mu(D)$

$\Rightarrow \; D^c \subseteq A \subseteq C.$ &

$$\mu_*(A) = \sup_{A \supset E \in \mathcal{A}} \mu(E) \geq \mu(D^c) = \mu(X) - \mu(D) = \mu^*(X) - (\mu(X) - \mu^*(A))$$

$$= \mu^*(A) \quad \boxed{\text{if } \mu(X) < \infty}$$

$\therefore \; \mu_*(A) = \mu^*(A).$

3. $\mathcal{A}_0 = \{ \bigcup_{j=1}^{k} (a_j, b_j] : 0 \leq a_1 \leq b_1 \leq \cdots \leq a_k \leq b_k \leq 1 \}.$

$\mu_0 : \mathcal{A}_0 \rightarrow [0, \infty)$ by

$$\mu_0\left(\bigcup_{j=1}^{k} (a_j, b_j] \right) = \sum_{j=1}^{k} (b_j - a_j)$$

(a). $\mathcal{A}_0$ is an algebra. : Assume that you are working on $X = (0, 1]$

half-open, closed segment.

then it is easy.

(b). (c). (d) : Easy.