1.3 - 8. $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$. then

① $\mu(\liminf E_j) \leq \liminf \mu(E_j)$.

pf) $\liminf E_j = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} E_j$ by definition,

$$\Rightarrow \mu(\liminf E_j) = \lim_{n\to\infty} \mu\left(\bigcap_{j=n}^{\infty} E_j\right)$$

$$= \liminf_{n\to\infty} \mu\left(\bigcap_{j=n}^{\infty} E_j\right) \leq \liminf_{n\to\infty} \mu(E_n)$$

② $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu\left(\bigcup_{1}^{\infty} E_j\right) < \infty$.

pf) $\limsup E_j = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j$

$$\mu(\limsup E_j) = \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j\right)$$

$$= \lim_{n\to\infty} \mu\left(\bigcup_{j=n}^{\infty} E_j\right)$$

$$= \limsup_{n\to\infty} \mu\left(\bigcup_{j=n}^{\infty} E_j\right) \geq \limsup_{n\to\infty} \mu(E_j).$$

9. $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

pf)

$E \cup F = E \cup (F \setminus E)$. and $E \cap (F \setminus E) = \emptyset$.

$$\therefore \mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F \setminus E) + \mu(E \cap F)$$

$$= \mu(E) + \mu(F) \quad \because F = (F \setminus E) \cup (E \cap F) \text{ and } (E \setminus F) \cap (E \cap F) = \emptyset.$$

10. Given $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$.

Then $\mu_E$ is a measure.

pf) (i) $\mu_E(\emptyset) = \mu(\emptyset \cap E) = \mu(\emptyset) = 0$.

(ii) let $\{E_j\}_{j=1}^{\infty}$ be a sequence of disjoint sets in $\mathcal{M}$.

Then
$$\mu_E\left(\bigcup_1^{\infty} E_j\right) = \mu\left(E \cap \left(\bigcup_1^{\infty} E_j\right)\right)$$
$$= \mu\left(\bigcup_1^{\infty} (E \cap E_j)\right)$$
$$= \sum_1^{\infty} \mu(E \cap E_j)$$
$$= \sum_1^{\infty} \mu_E(E_j).$$

1-3. 1. Given $\mu$ on $\mathcal{A}$ : $\sigma$-algebra. Define,

$\mu^* : P(X) \to [0, \infty]$ by
$$\mu^* E = \inf_{\mathcal{A} \ni A \supset E} \mu A$$

Show the following.

(a) $\mu^*$ is an outer measure.

① $\mu^* \emptyset = \inf_{A \supset \emptyset} \mu A = \mu(\emptyset) = 0$

② If $E \subset F \Rightarrow \{A \in \mathcal{A} : F \subset A\} \subseteq \{B \in \mathcal{A} : E \subset B\}$

$\Rightarrow \inf_{E \subset B} \mu B \le \inf_{F \subset A} \mu A$

$\Rightarrow \mu^* E \le \mu^* F$

③ $\therefore E_j \subseteq X$.

For any $\varepsilon > 0$, there exists $A_j \in \mathcal{A}$ such that $E_j \subseteq A_j$

and $\mu(A_j) \le \mu^*(E_j) + \frac{\varepsilon}{2^j}$.

$\Rightarrow \bigcup_j E_j \subseteq \bigcup_j A_j$ and.

$$\mu^*\left(\bigcup_j E_j\right) \le \mu\left(\bigcup_j A_j\right) \le \sum_j \mu(A_j) \le \sum_j \left[\mu^*(E_j) + \frac{\varepsilon}{2^j}\right]$$

$$= \left(\sum_j \mu^*(E_j)\right) + \varepsilon.$$

Since $\varepsilon$ is arbitrary, we have

$$\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \le \sum_j \mu^*(E_j)$$

b) $\mu^*(Z) = 0$ iff $\exists A \in \mathcal{A}$ with $Z \subset A$ and $\mu A = 0$

($\Leftarrow$) Trivial

($\Rightarrow$) Suppose $\mu^*(Z) = \inf_{Z \subseteq A \in \mathcal{A}} \mu A = 0$,

Then there exist $A_n \in \mathcal{A}$ s.t. $Z \subseteq A_n$ and $\mu(A_n) \le \frac{1}{n}$

let $B = \bigcap_{n=1}^{\infty} A_n$, then without loss of generality,

we can assume $A_n$ is decreasing, and $Z \subset B$.

$$\therefore \mu^*(Z) = \inf_{Z \subseteq A \in \mathcal{A}} \mu A \le \mu(B) = \lim_{n\to\infty} \mu(A_n) \le \lim_{n\to\infty} \frac{1}{n} = 0$$

Here $B = A$.

(c). $\bar{\mathcal{A}} = \{ A \cup Z : A \in \mathcal{A},\ \mu^* Z = 0 \}$, $\bar{\mu} : \bar{\mathcal{A}} \to [0,\infty]$ by

$$\bar{\mu}(A \cup Z) = \mu A.$$

Show that $\bar{\mathcal{A}}$ is a $\sigma$-algebra and $\bar{\mu}$ is a measure.

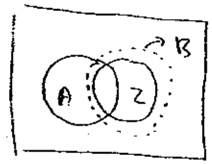
pf) (i) $\emptyset,\ X \in \bar{\mathcal{A}}$ $\quad (\because\ \mathcal{A} \subseteq \bar{\mathcal{A}})$.

(ii) $(A \cup Z)^c = ?$

From (b) There exist $B \in \mathcal{A}$ with $Z \subseteq B$ and $\mu(B) = 0$

$\Rightarrow$

$$(A \cup Z)^c = (A \cup B)^c \cup [B \setminus (A \cup Z)]$$




Here $(A \cup B)^c \in \mathcal{A}$ and

$[B \setminus (A \cup Z)] \subseteq B. \Rightarrow \mu^* [B \setminus (A \cup Z)] = 0.$

$\therefore\ (A \cup Z)^c \in \bar{\mathcal{A}}.$

(iii). $E_n \in \bar{\mathcal{A}} \Rightarrow E_n = A_n \cup Z_n.$

$$\Rightarrow \bigcup_{n=1}^{\infty} E_n = \left( \bigcup_{n=1}^{\infty} A_n \right) \cup \left( \bigcup_{n=1}^{\infty} Z_n \right) \in \bar{\mathcal{A}}.$$

$\therefore\ \bar{\mathcal{A}}$ is an $\sigma$-algebra. //

$\bar{\mu}$: First we need to prove $\bar{\mu}$ is well-defined.

let $E = A_1 \cup Z_1 = A_2 \cup Z_2$. for some $A_1, A_2 \in \mathcal{A}$ and $\mu^*(Z_1) = \mu^*(Z_2) = 0$.

then $\exists\ B_1, B_2 \in \mathcal{A}$ such that $Z_1 \subset B_1$, $Z_2 \subset B_2$ & $\mu(B_1) = \mu(B_2) = 0$

$A_1 \subseteq E \subseteq A_1 \cup B_1$, $A_2 \subseteq E \subseteq A_2 \cup B_2 \Rightarrow \mu(A_1) = \mu(A_2) = \bar{\mu}(E).$

ii) $\bar{\mu}(\emptyset)=0$.

iii) let $\{E_n\}$ be a seq of disjoint sets in $\bar{A}$.

$\Rightarrow$ $E_n = A_n \cup Z_n$.

$\Rightarrow$ $\bigcup_n E_n = \left(\bigcup_n A_n\right) \cup \left(\bigcup_n Z_n\right)$

$\Rightarrow$ $\bar{\mu}\left(\bigcup_n E_n\right) = \mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n) = \sum_n \bar{\mu}(E_n)$.

$\therefore$ $\bar{\mu}$ is a measure

1.4-1. $B \subset P(X)$ : closed under complements and finite unions and is also closed under countable disjoint union

Prove that $B$ is a $\sigma$-algebra.

pf) It is enough to show that $B$ is closed under countable union.

let $B_n \in \mathcal{B}$.

define
$$E_1 = B_1$$
$$E_2 = B_2 \setminus B_1$$
$$\vdots$$
$$E_n = B_n \setminus \bigcup_{i=1}^{n-1} B_i$$
$$\vdots$$

then $\bigcup_n B_n = \bigcup_n E_n \in \mathcal{B}$ ($\because$ $E_n$ are disjoint)