32. $\mu(X) < \infty$. define $\rho(f,g) = \int \frac{|f-g|}{1+|f-g|} \, d\mu$.

Then ρ is a metric, and $f_n \to f$ w.r.t the metric iff $f_n \to f$ measure

pf). lemma. If $0 \le a \le b$, then $\frac{a}{1+a} \le \frac{b}{1+b}$.

pf). $ab + a \le ab + b \Rightarrow a(1+b) \le b(1+a) \Rightarrow \frac{a}{1+a} \le \frac{b}{1+b}$. //

$\forall f, g, h$

$$\frac{|f-h|}{1+|f-h|} = \frac{|f-g+g-h|}{1+|f-g+g-h|} \le \frac{|f-g|+|g-h|}{1+|f-g|+|g-h|}$$

$$= \frac{|f-g|}{1+|f-g|+|g-h|} + \frac{|g-h|}{1+|f-g|+|g-h|}$$

$$\le \frac{|f-g|}{1+|f-g|} + \frac{|g-h|}{1+|g-h|}.$$

$\Rightarrow$ Triangle inequality for ρ is done, and other properties of distance are easy.

$\therefore$ ρ is distance.

$\therefore$ $\rho(f_n, f) \to 0$ iff $f_n \to f$ in measure.

$\Rightarrow$) Suppose $f_n \not\to f$ in measure.

then $\exists$ a subsequence $\{f_{n_k}\}$ and positive numbers $a, b > 0$. such that

$$\mu(|f_{n_k} - f| > a) > b \quad \text{for all } n_k$$

$$\Rightarrow \quad \rho(f_{n_k}, f) = \int \frac{|f_{n_k}-f|}{1+|f_{n_k}-f|}\,d\mu \geq \frac{a}{1+a}\cdot b \not\to 0.$$

contradiction.

$\therefore$ $f_n \to f$ in measure.

($\Leftarrow$).

$$\rho(f_n, f) = \int \frac{|f_n-f|}{1+|f_n-f|}\,d\mu = \int_{|f_n-f|>\varepsilon} \frac{|f_n-f|}{1+|f_n-f|}\,d\mu + \int_{|f_n-f|<\varepsilon} \frac{|f_n-f|}{1+|f_n-f|}\,d\mu$$

$$\leq \mu(|f_n-f|>\varepsilon) + \frac{\varepsilon}{1+\varepsilon}\cdot \mu(|f_n-f|<\varepsilon)$$

$$\leq \underbrace{\mu(|f_n-f|>\varepsilon) + \frac{\varepsilon}{1+\varepsilon}\,\mu(X)}_{\text{arbitrarily small as } n\to\infty}.$$

$\therefore$ $\rho(f_n, f) \to 0.$ //

33. If $f_n \geq 0$ and $f_n \to f$ in measure, then $\int f \leq \liminf \int f_n$.

pf). Suppose $\exists$ a subseq $\{f_{n_k}\}$ such that

$$\int f_{n_k} < \int f - \varepsilon. \quad \text{for some } \varepsilon > 0. \quad -①$$

Since $f_{n_k} \to f$ in measure, (by Thm 2.30) $\exists$ a subseq $\{f_{n_{k_i}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_i}} \to f$ a.e.

Fatou's lemma says,

$$\int f \leq \liminf \int f_{n_{k_i}} \quad - ②$$

But ②. is contradicted with ①.

$\therefore$ $\int f \leq \liminf \int f_n$ //

34. $|f_n| \le g \in L^1$ and $f_n \to f$ in measure.

(a) $\int f = \lim \int f_n$.

pf) $g - f_n \ge 0 \Rightarrow \cancel{\int g} - f \le \liminf \int g - f_n = \cancel{\int g} - \limsup \int f_n$.

$\therefore \int f \ge \limsup \int f_n$.

$f_n - g \ge 0 \Rightarrow \int f - g \le \liminf \int f_n - g = \liminf \int f_n - \int g$.

$\therefore \int f \le \liminf \int f_n$.

$\therefore \int f = \lim \int f_n$. //

(b) $f_n \to f$ in L^1.

pf)

Let $\{f_{n_k}\}$ be an arbitrary subsequence of $\{f_n\}$, then

$\exists$ a subseq $\{f_{n_{k_i}}\}$ of $\{f_{n_k}\}$ such that $f_{n_{k_i}} \to f$ a.e.

Since $|f_{n_{k_i}}| \le g \in L^1$. by LDCT.

$f_{n_{k_i}} \to f$ in L^1.

Since $\{f_{n_k}\}$ is an arbitrary subseq of $\{f_n\}$,

$f_n \to f$ in L^1.

35. $f_n \to f$ iff $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ $\ni$

$$\mu(|f_n - f| \geq \varepsilon) < \varepsilon \quad \text{for} \quad n \geq N.$$

($\Rightarrow$). Since $\mu(|f_n - f| \geq \varepsilon) \to 0$ as $n \to \infty$, $\exists$ $N \in \mathbb{N}$. $\ni$.

$$\mu(|f_n - f| \geq \varepsilon) < \varepsilon \quad \text{for all} \quad n \geq N.$$

($\Leftarrow$). Suppose, $f_n \not\to f$ in measure, then $\exists$ $\varepsilon > 0$.

$\ni$.

$$\mu(|f_n - f| > \varepsilon) \not\to 0.$$

$\Rightarrow$ $\exists$ a subseq f_{n_k}, and a positive number $\delta > 0$

$\ni$.

$$\mu(|f_{n_k} - f| > \varepsilon) \geq \delta \qquad \text{for all} \quad n_k.$$

① $\underline{\delta > \varepsilon}$

$$\mu(|f_{n_k} - f| > \varepsilon) \geq \delta > \varepsilon \quad \Rightarrow \quad \mu(|f_{n_k} - f| > \varepsilon) > \varepsilon.$$

② $\underline{\delta < \varepsilon}$

$$\mu(|f_{n_k} - f| > \delta) \geq \mu(|f_{n_k} - f| > \varepsilon) \geq \delta.$$

$$\Rightarrow \quad \mu(|f_{n_k} - f| > \delta) \geq \delta.$$

In any case, we have a contradiction,

$\therefore$ $f_n \to f$ in measure.

39. $f_n \to f$ almost uniformly $\Rightarrow$ $f_n \to f$ a.e and in measure.

pf) Convergence in measure is Trivial.

let $A = \{ x : f_n(x) \not\to f(x) \}$,

Suppose $\mu(A) > 0$, then $\exists$ $B \subset A$ s.t. $\mu(B) < \frac{\mu(A)}{2}$ and

$f_n \to f$ uniformly on $A \setminus B$. and $\mu(A \setminus B) > \frac{\mu(A)}{2} > 0$.

but $f_n \not\to f$ pointwise on $A \setminus B \subseteq A$.

Contradiction $\Rightarrow$ $\mu(A) = 0$.

$\therefore$ $f_n \to f$ a.e.

49. $(X_j, \mathcal{M}_j)$ is a measurable space for $j=1,2,3$. then

① $\bigotimes_1^3 \mathcal{M}_j = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$

② If μ_j is a σ-finite measure on $(X_j, \mathcal{M}_j)$, then
$\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$.

Hint ① Show that $(\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$ is a σ-algebra generated by
$\mathcal{A} = \{ A_1 \times A_2 \times A_3 ; A_j \in \mathcal{M}_j \}$. - rectangles.

② Show that, $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$ on $\mathcal{A}$.

50. $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f \in L^+(X)$.
Let $G_f = \{ (x, y) \in X \times [0, \infty] ; y \le f(x) \}$.

$\Rightarrow$ ① G_f is $\mathcal{M} \times \mathcal{B}_{\mathbb{R}}$ - measurable and $\mu \times m(G_f) = \int f \, d\mu$:

② The same is also true if $y \le f(x)$ is replaced by $y < f(x)$.

pf) Define $g : X \times [0, \infty] \to [-\infty, \infty]$ by
$g(x, y) = f(x) - y$.

$\Rightarrow$ g is measurable.

[Remark : g is not well-defined because if $f(x) = \infty$ & $y = \infty$.
then $\infty - \infty$?.
So you need to do some work, but idea is same]

Then $\quad G_f = g^{-1}([0,\infty])$; measurable set in $X \times [0,\infty]$.

roughly.

and $\quad \mu \times m(G_f) = \int \chi_{G_f} \, d(\mu \times m) = \int_X m(G_x) \, d\mu(x)$

$$= \int_X \int_0^{f(x)} 1 \, dm \, d\mu(x)$$

$$= \int_X f(x) \, d\mu$$

Same is true if $y \le f(x)$ is replaced by $y < f(x)$.