1. Let $\{f_n = f_n^r + i f_n^c\}$ be a sequence in L^1 such that

(a) $f_n \to f$ a.e

(b) $\exists$ a nonnegative $g \in L^1$ such that $|f_n| \le g$. a.e for all n.

Then $f \in L^1$ and $\int f = \lim \int f_n$.

(pf) Since $f_n \to f$ a.e, $f = f^r + i f^c$ such that

$$f_n^r \to f^r, \quad f_n^c \to f^c \quad \text{a.e.}$$

and $|f_n^r|, |f_n^c| \le |f_n| \le g \in L^1$.

By the dominated convergence Theorem for real valued functions,

$f^r, f^c \in L^1$ and

$$\int f^r = \lim \int f_n^r$$

$$\int f^c = \lim \int f_n^c.$$

$\Rightarrow$ $f = f^r + i f^c \in L^1$ and $\int f = \lim \int f_n$ //

2. Show that $\widetilde{M} = \{A \subset X_\alpha : \pi_\alpha^{-1} A \in M(\mathcal{E}_0)\}$ is a σ-algebra.

(pf) (i) $\emptyset \in \widetilde{M}$ trivial.

(ii) If $A \in \widetilde{M} \Rightarrow \pi_\alpha^{-1}(A) \in M(\mathcal{E}_0)$

and $(\pi_\alpha^{-1}(A))^c \in M(\mathcal{E}_0)$ $\because$ $M(\mathcal{E}_0)$ is a σ-algebra,

but $\pi_\alpha^{-1}(A^c) = (\pi_\alpha^{-1}(A))^c \in M(\mathcal{E}_0)$

$\therefore$ $A^c \in \widetilde{M}$.

(iii) If $\{A_n\}_{n=1}^{\infty} \subseteq \widetilde{M} \Rightarrow \pi_\alpha^{-1}(A_n) \in M(\mathcal{E}_0)$ for all n.

$\Rightarrow \bigcup_{n=1}^{\infty} \pi_\alpha^{-1}(A_n) \in M(\mathcal{E}_0)$

$\Rightarrow \pi_\alpha^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \pi_\alpha^{-1}(A_n) \in M(\mathcal{E}_0)$

$\therefore$ $\bigcup_{n=1}^{\infty} A_n \in \widetilde{M}$.

$\therefore$ $\widetilde{M}$ is a σ-algebra for all α.

3. Every separable metric space is second countable; $\mathbb{R}^n$ is second countable

pf) let (X, d) be a separable metric space. and $\{a_n\}$ be a countable dense subset of X.

Define. $B_{n,k} = \{ x \in X ; d(x, a_n) < \frac{1}{k} \}$.

Then $B_{n,k}$ is trivially open for all n, k.

and $B_{n,k}$ is a base.

pf) let V be an open set.

$\forall$ $a \in V$, $\exists$ $\delta > 0$ s.t. $B_\delta(a) = \{ x \in X ; d(x,a) < \delta \} \subseteq V$

Since $\{a_n\}$ is dense, $\exists$ n_0 s.t. $d(a, a_{n_0}) < \frac{\delta}{4}$.

then $\exists$ k s.t. $\frac{\delta}{4} < \frac{1}{k} < \frac{\delta}{2}$. for $\delta < 1$.

$\Rightarrow$ $a \in B_{n_0, k} \subseteq V$

$\therefore$ $B_{n,k}$ is a Base for (X, d) and Countable.//

$\mathbb{R}^n$; $\mathbb{Q}^n = \{ \bar{z} \in \mathbb{R}^n ; \bar{z} = (z_1, \cdots, z_n)$ z_i is rational number $\}$.

is a countable dense subset of $\mathbb{R}^n$.

4. $\widetilde{\mathcal{M}} = \{ \widetilde{E} \subset X_1 \times X_2 : \{x \in X_1 : (x_1, x_2) \in \widetilde{E}\} \in \mathcal{M}_1 \text{ for all } x_2 \in X_2 \}$.

Show that $\widetilde{\mathcal{M}}$ is a σ-algebra containing all rectangles.

pf) (i) $X_1 \times X_2 \in \widetilde{\mathcal{M}} \Rightarrow \widetilde{\mathcal{M}} \neq \emptyset$.

(ii) If $\widetilde{E} \in \widetilde{\mathcal{M}} \Rightarrow \{x \in X_1 : (x_1, x_2) \in \widetilde{E}\} \in \mathcal{M}$ for all $x_2 \in X_2$.

$\Rightarrow \{x \in X_1 : (x_1, x_2) \in \widetilde{E}\}^c \in \mathcal{M}$ $\quad \mathcal{M}$ is σ-algebra

$\Rightarrow \{x \in X_1 : (x_1, x_2) \in \widetilde{E}^c\} \in \mathcal{M}$.

$\therefore \widetilde{E}^c \in \widetilde{\mathcal{M}}$.

(iii) let $\{\widetilde{E}_n\} \subseteq \widetilde{\mathcal{M}}$. $\Rightarrow \{x \in X_1 : (x_1, x_2) \in \widetilde{E}_n\} \in \mathcal{M}$ for all n,

for fixed x_2.

$\Rightarrow \bigcup_{n=1}^{\infty} \{x \in X_1 : (x_1, x_2) \in \widetilde{E}_n\} \in \mathcal{M}$.

$\Rightarrow \{x \in X_1 : (x_1, x_2) \in \bigcup_{n=1}^{\infty} \widetilde{E}_n\} \in \mathcal{M}$.

Since x_2 is arbitrary, $\bigcup_{n=1}^{\infty} \widetilde{E}_n \in \widetilde{\mathcal{M}}$.

$\therefore$ $\widetilde{\mathcal{M}}$ is a σ-algebra.

(iv) If $E_1 \in \mathcal{M}_1$, $E_2 \in \mathcal{M}_2$, then.

$$\{x_1 \in X : (x_1, x_2) \in E_1 \times E_2\} = \begin{cases} E_1 & \text{if } x_2 \in E_2. \\ \emptyset & \text{if } x_2 \notin E_2. \end{cases}$$

$\therefore E_1 \times E_2 \in \widetilde{\mathcal{M}}$

46. $X = Y = [0,1]$. $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$. , μ = Lebesgue measure.

ν = counting measure.

$D = \{(x,x) : x \in [0,1]\}$.

then, $\iint \chi_D \, d\mu \, d\nu$, $\iint \chi_D \, d\nu \, d\mu$ and $\int \chi_D \, d(\mu \times \nu)$ are all unequal.

(i) $\iint \chi_D \, d\mu \, d\nu = \int_0^1 \mu(\{x\}) \, dx = 0$.

(ii) $\iint \chi_D \, d\nu \, d\mu = \int_0^1 \nu(\{x\}) \, dx = \int_0^1 1 \, dx = 1$.

(iii) $\iint \chi_D \, d(\mu \times \nu) = \mu \times \nu(D) = \inf \left\{ \sum_1^\infty \mu(A_i) \times \nu(B_i) : A_i, B_i \in \mathcal{B}_{[0,1]}, D \subseteq \bigcup_1^\infty A_i \times B_i \right\}$.

Since $[0,1]$ is an uncountable set, at least one B_i is uncountable with $\mu(A_i) \neq 0$.

$\Rightarrow \mu(A_i) \times \mu(B_i) = \infty$.

$\therefore \iint \chi_D \, d(\mu \times \nu) = \mu \times \nu(D) = \inf \left\{ \sum \mu(A_i) \times \mu(B_i) : D \subseteq \bigcup A_i \times B_i \right\}$

$= \inf \{\infty\} = \infty$.

//