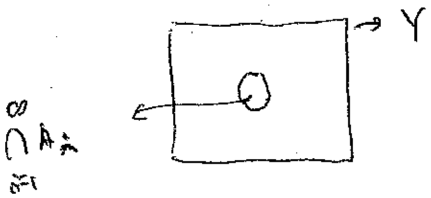
1. a let $A_i \in \mathcal{R}$. $i = 1, 2, 3, \cdots$

then, $$\left(\bigcup_{i=1}^{\infty} A_i\right) \setminus \left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcup_{j=1}^{\infty} \left[\left(\bigcup_{i=1}^{\infty} A_i\right) \setminus A_j\right]. \quad - ①$$

⌈ the way you think ① is this.

let $Y = \bigcup_{i=1}^{\infty} A_i$ be a whole set,



$$\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c \quad \text{(D' morgan's rule)}. \quad - ②$$

Here $A^c = Y \setminus A$.

① & ② are equivalent. ⌋

$\therefore$ From ①. $$\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i\right) \setminus \bigcup_{j=1}^{\infty} \left[\left(\bigcup_{i=1}^{\infty} A_i\right) \setminus A_j\right] \in \mathcal{R}.$$

b. $\mathcal{R}$ is a ring (σ- ring).

$\mathcal{R}$ is an algebra (σ- algebra) $\Longleftrightarrow$ $X \in \mathcal{R}$.

($\Rightarrow$) Trivial

($\Leftarrow$) If $X \in \mathcal{R}$, then $\forall E \in \mathcal{R}$, $X \setminus E \in \mathcal{R}$

$\therefore$ Algebra.

c. $\mathcal{R}$ is a σ- ring. Then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.

let $\mathcal{A} = \{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$.

(i) $\emptyset, X \in \mathcal{A}$; trivial

(ii) If $E \in \mathcal{A}$ then $E^c \in \mathcal{A}$; trivial.

(iii). let $E_i \in \mathcal{A}$ $\Rightarrow$ either $E_i \in \mathcal{A}$ or $E_i^c \in \mathcal{A}$.

$$\bigcup_{i=1}^{\infty} E_i = \left(\bigcup_{i\in I} E_i\right) \cup \left(\bigcup_{i \in J} E_i\right) \qquad I = \{ i \mid E_i \in \mathcal{R}\}. \quad J = \{ i : E_i^c \in \mathcal{R}\}.$$

$$= \left(\bigcup_{i\in I} E_i\right) \cup \Big(\underbrace{\bigcap_{i \in J} E_i^c}_{\in \mathcal{R} \text{ from } ⓐ}\Big)^c \in \mathcal{R}.$$

d. $\mathcal{R}$ is a σ-ring then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra

let $\mathcal{A} = \{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$.

(i) $\emptyset, X \in \mathcal{A}$

(ii) $E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A}$ $\Big\}$ trivial.

(iii) let $E_i \in \mathcal{A}$.

then $\left(\bigcup_{i=1}^{\infty} E_i\right) \cap F = \bigcup_{i=1}^{\infty} (E_i \cap F) \in \mathcal{R}$.

$\therefore \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$. //

2. Easy.

3. Let $\mathcal{M}$ be an infinite σ-algebra.

(a) $\mathcal{M}$ contains an infinite sequence of disjoint sets.

(pf). Since $\mathcal{M}$ is an infinite σ-algebra, there exists $A_1 \in \mathcal{M}$, such that

(i) $A_1 \neq \emptyset$, $A_1 \neq X$.

(ii) $\{A_1 \cap E : E \in \mathcal{M}\}$ is infinite or

$\{A_1^c \cap E \mid E \in \mathcal{M}\}$ is infinite.

Suppose $\mathcal{M}_1 = \{A_1 \cap E \mid E \in \mathcal{M}\}$ is infinite.

then there exists $A_2 \in \mathcal{M}_1$ such that

(i) $A_2 \neq \emptyset$, $A_2 \neq A_1$.

(ii) $\{A_2 \cap E : E \in \mathcal{M}_1\}$ is infinite

$\Rightarrow$ $A_2 \subset A_1$ and $A_2 \neq A_1$.

In this way, we can construct A_i such that.

$\cdots \subset A_{i+1} \subset A_i \subset A_{i-1} \subset \cdots \subset A_1$ and $A_i \neq \emptyset$

$A_{i+1} \neq A_i$.

let $B_1 = A_1 \setminus A_2$, $B_2 = A_2 \setminus A_3$, $\cdots$

Then $B_i \neq \emptyset$ and $B_i \cap B_j = \emptyset$ if $i \neq j$. //

4. An algebra $\mathcal{A}$ is a σ-algebra iff $\mathcal{A}$ is closed under countable increasing union.

($\Rightarrow$) Trivial from definition of σ-algebra.

($\Leftarrow$) It is enough to show,

$$\forall \; E_i \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}.$$

let $B_1 = E_1$,

$B_2 = E_1 \cup E_2$

$\vdots$

$B_n = \bigcup_{i=1}^{n} E_i$.

then $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} B_i \in \mathcal{A}$. //

5. $\mathcal{M} = \mathcal{M}(\mathcal{E})$, then $\mathcal{M} = \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$

$\mathcal{F}$: countable subset of $\mathcal{E}$.

(pf) It is enough to show that $\bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ is a σ-algebra.

(i) $\emptyset, X \in \mathcal{M}(\mathcal{F})$ for all $\mathcal{F}$ $\Rightarrow$ $\emptyset, X \in \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$

(ii) $A \in \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$ $\Rightarrow$ $A^c \in \bigcup_{\mathcal{F}} \mathcal{M}(\mathcal{F})$: trivial.

(ii). $E_i \in \bigcup_{F_i}^{\infty} M(F)$, $\Rightarrow$ $\exists$ $E_i \in M(F_i)$ for some F_i.

let $G = \bigcup_{i=1}^{\infty} F_i$; countable $\because$ Each F_i is countable

$\therefore$ $E_i \in M(F_i) \subset M(G)$. for all $i = 1, 2, \cdots n, \cdots$

$\Rightarrow$ $\bigcup_{i=1}^{\infty} E_i \in M(G)$. and G is countable.

$\Rightarrow$ $\bigcup_{i=1}^{\infty} E_i \in M(G) \subseteq \bigcup_{F} M(F)$. //.