1. Show that if μ is a signed measure and $0 < \mu E < \infty$, then E contains a positive set. (Hint: Use the Hahn decomposition directly.)

2. We have proved the Lebesgue-Radon-Nikodym Theorem (Theorem 3.8) for nonnegative finite measures. Complete the following to verify the theorem for nonnegative σ-finite measures.

   (a) Let μ and ν be (nonnegative) σ-finite measures. There exist disjoint sets $X_1, X_2, \ldots$ such that
   $$\nu(X_j), \mu(X_j) < \infty \qquad \forall\, j.$$

   (b) Apply the theorem to $\nu_j E := \nu(E \cap X_j)$ and $\mu_j E := \mu(E \cap X_j)$ to obtain Lebesgue decompositions
   $$\nu_j = (\nu_j)^{\perp} + (\nu_j)_0$$
   and Radon-Nikodym derivatives
   $$(\nu_j)_0 E = \int_E f_j.$$
   Show that one may assume the following:

      i. $f_j : X \to [0, \infty)$,

      ii. $f_j|_{X_j^c} \equiv 0$,

      iii. $f_j \in L^1$.

   Consider $\nu^{\perp} = \sum(\nu_j)^{\perp}$, $\nu_0 = \sum(\nu_j)_0$, and $f = \sum f_j$. ($\nu^{\perp}$ and ν_0 are nonnegative measures; f is a nonnegative measurable function.)

   (c) Show that $\nu = \nu^{\perp} + \nu_0$.

   (d) Show that $\nu^{\perp} \perp \mu$.

   (e) Show that $\nu_0 \ll \mu$.

   From (c–e) we see that ν has a Lebesgue decomposition.

   (f) Show that $\nu_0 E = \int_E f$, so that the existence of the Radon-Nikodym derivative holds as well.

   (g) (Uniqueness of $\nu^{\perp}$ and ν_0) Let $\nu = \lambda^{\perp} + \lambda_0$ give another Lebesgue decomposition of ν. Set
   $$(\lambda^{\perp})_j E := \lambda^{\perp}(E \cap X_j) \quad \text{and} \quad (\lambda_0)_j E := \lambda_0(E \cap X_j).$$
   (These are finite nonnegative measures.) Show that $\nu_j = (\lambda^{\perp})_j + (\lambda_0)_j$ gives a Lebesgue decomposition of ν_j. Thus, by the uniqueness in the finite case, $(\nu_j)^{\perp} = (\lambda^{\perp})_j$ and $(\nu_j)_0 = (\lambda_0)_j$. Show from this that $\lambda^{\perp} = \nu^{\perp}$ and $\lambda_0 = \nu_0$.

   (h) (Uniqueness of f) Assume $\nu_0 E = \int_E g$ for some $g \geq 0$. Define $(\nu^{\perp})_j E := \nu^{\perp}(E \cap X_j)$ and $(\nu_0)_j E := \nu_0(E \cap X_j) = \int_E g\chi_{X_j}$. Show that $(\nu^{\perp})_j = (\nu_j)^{\perp}$ which is unique and $(\nu_0)_j = (\nu_j)_0$ which is also unique. Conclude from the uniqueness of f_j that $g\chi_{X_j} = f_j$. Show that $g = \sum g\chi_{X_j} = \sum f_j = f$ a.e. $[\mu]$.

3. In the problem above one proves the Lebesgue-Radon-Nikodym theorem for nonnegative σ-finite measures. Complete the following verifying the theorem for σ-finite measures μ and ν with ν a signed measure.

   (a) ν^+ and ν^- are nonnegative σ-finite measures. (They are also orthogonal.) We have, therefore, unique Lebesgue decompositions

   $$\nu^+ = (\nu^+)^\perp + (\nu^+)_0$$

   and

   $$\nu^- = (\nu^-)^\perp + (\nu^-)_0.$$

   Show that

   $$\nu = \left[(\nu^+)^\perp - (\nu^-)\perp\right] + \left[(\nu^+)_0 - (\nu^-)_0\right]$$

   is a Lebesgue decomposition for ν.

   (b) According to the previous part (a), set

   $$\nu^\perp = (\nu^+)^\perp - (\nu^-)^\perp$$

   $$\nu_0 = (\nu^+)_0 - (\nu^-)_0.$$

   From the nonnegative case, $\exists\ f, g \geq 0$ s.t.

   $$(\nu^+)_0 E = \int_E f$$

   and

   $$(\nu^-)_0 E = \int_E g.$$

   Show that $\phi = f - g$ is a Radon-Nikodym derivative for ν_0. (You also need to explain why $\phi^- \in L^1$.)

   (c) (Uniqueness of $\nu^\perp$ and ν_0) Let $\nu = \lambda^\perp + \lambda_0$ be another Lebesgue decomposition for ν.

   Let $X = \Lambda \uplus M$ be Lebesgue decomposition sets for $\lambda^\perp$ and μ, i.e., M is null for $\lambda^\perp$ and $\mu\Lambda = 0$.

      i. Show that $\lambda^\perp \perp \lambda_0$ and Λ is null for Λ_0.

      ii. Show that Λ is null for $(\lambda_0)^\pm$ and M is null for $(\lambda^\perp)^\pm$ (n.b. Ex. 3.1.2)

      iii. Show that the Jordan decomposition of ν is given by

      $$\nu^\pm = (\lambda^\perp)^\pm + (\lambda_0)^\pm. \tag{1}$$

      (Hint: This is a bit tricky — I think. Let $X = A \uplus B = A^\perp \uplus B^\perp = A_0 \uplus B_0$ be the Hahn decompositions for ν, $\lambda^\perp$, and λ_0 respectively; show $\left[(\lambda^\perp)^+ + (\lambda_0)^+\right] A = 0$ by contradiction using the fact that

      $$\begin{aligned}\left[(\lambda^\perp)^+ + (\lambda_0)^\perp\right] A &= (\lambda^\perp)^+(A \cap \Lambda) + (\lambda_0)^+(A \cap M) \\ &= (\lambda^\perp)^+(A \cap \Lambda \cap B^\perp) + (\lambda_0)^+(A \cap M \cap B_0).)\end{aligned}$$

iv. Show that the Lebesgue decompositions for $\nu^{\pm}$ are given in (1).

v. Use the uniqueness of the Lebesgue decompositions in (1) to conclude that $(\nu^{\pm})^{\perp} = (\lambda^{\perp})^{\pm}$ and $(\nu^{\pm})_0 = (\lambda_0)^{\pm}$.

vi. Show that $\lambda^{\perp} = \nu^{\perp}$ and $\lambda_0 = \nu_0$.

(d) (Uniqueness of Radon-Nikodym derivative) Use $\phi \equiv f - g$ from 3(b) above and assume ψ satisfies

$$\psi^- \in L^1$$
$$\nu_0 E = \int_E \psi = \int_E \psi^+ - \int_E \psi^- \qquad \forall\, E.$$

i. Show that the Lebesgue decompositions of $\nu^{\pm}$ are given by

$$\nu^{\pm} E = (\nu^{\pm})^{\perp} E + \int_E \psi^{\pm}.$$

ii. Use the uniqueness of Lebesgue decompositions in the nonnegative case to conclude that

$$\int_E f = \int_E \psi^- \qquad \forall\, E$$
$$\int_E g = \int_E \psi^+ \qquad \forall\, E.$$

iii. Use the conclusion of (ii) to show $\psi = \phi$ a.e. $[\mu]$.