1. Denote by $\mathcal{T}$ the standard topology on $\mathbb{R}$. and by $\mathcal{B}$ the Borel sigma algebra on $\mathbb{R}$.

   (a) Show that $\{U \subset [-\infty, \infty] : U \cap \mathbb{R} \in \mathcal{T}\}$ is a topology on $[-\infty, \infty]$.

   (b) Show that $\mathcal{B}_{[-\infty,\infty]} = \{U \subset [-\infty, \infty] : U \cap \mathbb{R} \in \mathcal{B}\}$ is the Borel σ-algebra on $[-\infty, \infty]$.

   (c) (Generalize Proposition 2.3) Let $(X, \mathcal{M})$ be a measure space. Show that the following are equivalent for $f : X \to [-\infty, \infty]$.

      i. $f^{-1}U \in \mathcal{M}$ for U open in $[-\infty, \infty]$.
      ii. $f^{-1}(a, \infty] \in \mathcal{M}$ for $a \in \mathbb{R}$.
      iii. $f^{-1}[a, \infty] \in \mathcal{M}$ for $a \in \mathbb{R}$.
      iv. $f^{-1}[-\infty, a) \in \mathcal{M}$ for $a \in \mathbb{R}$.
      v. $f^{-1}[-\infty, a] \in \mathcal{M}$ for $a \in \mathbb{R}$.

2. (Simple Functions) Let $(X, \mathcal{M})$ be a measure space. Recall that a *simple function* on X is defined to be any function of the form

$$f = \sum_{j=1}^{k} a_j \chi_{E_j} \tag{1}$$

   where the $a_j \in \mathbb{R}$ and the E_j are in $\mathcal{M}$.

   (a) Give an example to show that the representation in (1) is not unique.

   (b) Show that any simple function has a *unique* representation of the same form as in (1) if it is also assumed that [1] the a_j are distinct, [2] the E_j are disjoint and [3] $\cup E_j = X$.

   (c) Show that any simple function has a unique representation of the same form as in (1) if it is also assumed that [1] the a_j are distinct *and nonzero* and [2] the E_j are disjoint.

3. (Integration of Nonnegative Simple Functions) Let f be a nonnegative simple function as in the previous problem and let μ be a measure on $\mathcal{M}$. We define the integral of f as

$$\int f := \sum_{j=1}^{k} a_j \mu E_j$$

   where the representation satisfies the hypotheses [1] $a_j > 0$ and $a_i \neq a_j$ for $i \neq j$ and [2] $E_i \cap E_j = \phi$ if $i \neq j$ of (2c).

   (a) Show that if $f = \sum_{i=1}^{\ell} b_i \chi_{A_i}$ is any representation of f as in (1) with $b_i > 0$, then

$$\int f = \sum_{i=1}^{\ell} b_i \mu A_i.$$

   (b) Why couldn't we define $\int f$ for nonnegative simple functions as in (1) without any assumption on the coefficients?