1. Let μ be a locally finite Borel measure on $\mathbb{R}$. Consider $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} -\mu(x,0] & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \mu(0,x] & \text{if } x > 0. \end{cases}$$

Show that g is nondecreasing and $\inf_{x>a} g(x) = g(a)$.

2. Extend the discussion of the measure m on $\mathbb{R}$ to construct Lebesgue-Stieltjes measure μ_g on $\mathbb{R}$:

   (a) Recall the definition of the algebra $\mathcal{A}_0 = \mathcal{A}_0^0 \cup \mathcal{A}_0^1$ from the previous problem set.
   Let $g : \mathbb{R} \to \mathbb{R}$ be any nondecreasing right-continuous function.
   Show that $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ by

$$\mu_0 E = \sum_{j=1}^{k} [g(b_j) - \lim_{x \searrow a_j} g(x)]$$

   if $E = \cup(a_j, b_j] \in \mathcal{A}_0^0$ and

$$\mu_0 E = \sum_{j=1}^{k-1} [g(b_j) - \lim_{x \searrow a_j} g(x)] + \lim_{x \nearrow \infty} g(x) - g(a_k)$$

   if $E = \cup^{k-1}(a_j, b_j] \cup (a_k, \infty) \in \mathcal{A}_0^1$, defines a premeasure.

   (b) Prove the rest of the Theorem: There is a unique (locally finite) Borel measure μ_g on $\mathbb{R}$ such that

$$\mu_g(a, b] = g(b) - g(a).$$

3. Prove that all the Lebesgue-Stieltjes measures are Borel Regular.