Name: ______________________

Math 6327 Real Analysis Exam 2(Takehome)

1. Let f be a nonnegative Lebesgue measurable function on $[0,1]$ which is bounded. Let $\epsilon > 0$.
   (a) Show that there is a sequence of continuous functions $\{f_j\} \subset C^0[0,1]$ which converges to f in measure.
   (b) Show that there is a sequence $\{f_j\} \subset C^0[0,1]$ and a subset $E \subset [0,1]$ with $mE \geq 1-\epsilon$ and such that f_j converges uniformly to f on E.
   (c) Show that there is a function $\tilde{f} \in C^0[0,1]$ with $m\{x : \tilde{f}(x) \neq f(x)\} < \epsilon$. (This is called Lusin's Theorem.)
2. Let $\{f_j\}$ be a sequence of nonnegative functions in $L^1(\mathbb{R})$ and $f \in L^1(\mathbb{R})$ with $f_j \to f$ almost everywhere. Show that if $\int f_j \to \int f$, then $\int_E f_j \to \int_E f$ for every measurable set E.
3. Show that the standard Cantor function on $[0,1]$ is continuous and of bounded variation, but is not absolutely continuous. (Hint: Use the Fundamental Theorem for the last assertion.)
4. Recall that the *distrubution function* associated to a Borel measure μ on $\mathbb{R}$ is defined by
$$M(x) := \mu(-\infty, x].$$
   (a) Show that μ is absolutely continuous with respect to Lebesgue measure if and only if M is an absolutely continuous function.
   (b) Show that if μ is absolutely continuous with respect to Lebesgue measure, then the Radon-Nikodym derivative is M'.
5. (Product Measures and Fubini's Theorem) Throughout this problem, we use Lebesgue measures.
   (a) If E is a measurable subset of $\mathbb{R}$, then $\{(x,y) : x-y \in E\}$ is a measurable set in $\mathbb{R}^2$.
   (b) If $f, g \in L^1(\mathbb{R})$, then show that the *convolution*
$$f \star g(y) = \int f(y-x)g(x)dx$$
   is well defined.
   (c) Show that $f \star g = g \star f$.
6. Let $\mathcal{N}$ be a finite dimensional normed space.

(a) Show that $\mathcal{N}^*$ is also finite dimensional.

(b) Show that weak convergence in $\mathcal{N}$ implies strong convergence.

7. Show that $L^2[0,1]$ is separable, i.e., has a countable dense subset. (Hint: See Exercise 5.5.62.)