1. The statement of this problem requires some correction. (See corrected exam.) In particular, part (c) is not possible if the lower semicontinuous function f is not bounded below, and this is quite possible if the domain of f is $(0,1)$ as indicated in the problem, e.g., $f(x) = -1/x$. In addition, part (e) implicitly, and part (f) explicitly, consider the domain of definition to be $[0,1]$.

   We therefore invoke the universal instruction: *Solve the given problems — unless the problem is incorrectly stated. If the problem is incorrectly stated, figure out the correct statement, and solve that problem.*

   Let us make a new definition: Given any subset $E \subset \mathbb{R}$, f is *lower semicontinuous at* $x_0 \in E$ if
   $$f(x_0) \le \liminf_{E \ni x \to x_0} f(x).$$
   (The definition of lim inf is:
   $$\liminf_{E \ni x \to x_0} f(x) = \lim_{\delta \to 0} \inf\{f(x) : x \in E,\, 0 < |x - x_0| < \delta\}.)$$

   (a) Recall that f is *continuous* at a point x_0 if f is defined in a neighborhood of x_0 and
   $$\lim_{x \to x_0} f(x) = f(x_0). \tag{1}$$
   (In the case x_0 is an endpoint of an interval on which f is defined, say $[x_0, x_0 + \delta)$, we assume $f(x) \equiv f(x_0)$ for $x < x_0$.) Assuming the domain of definition is $(0,1)$ as given in the problem, there is no question about being defined in a neighborhood of each point, and f is continuous at x_0
   $$\begin{aligned} &\Leftrightarrow \liminf_{x \to x_0} f(x) = f(x_0) = \limsup_{x \to x_0} f(x) \qquad (2)\\ &\Leftrightarrow \liminf_{x \to x_0} f(x) \ge f(x_0) \text{ and } f(x_0) \ge \limsup_{x \to x_0} f(x)\\ &\Leftrightarrow f \text{ is lower and upper semicontinuous.} \end{aligned}$$

   (b) By Proposition 2.3 it is enough to show that $\{x \in (0,1) : f(x) > a\}$ is a Borel set whenever $a \in \mathbb{R}$. In fact, this set is open: Assume $f(x_0) > a$. We claim that $f(x) > a$ for x in some ball $Br(x_0)$. Otherwise, there exists a sequence $x_j \to x_0$ s.t. $f(x_j) \le a$. Thus, by semicontinuity $f(x_0) \le \liminf f(x_j) \le a$, which contradicts the fact that $f(x_0) > a$. Thus, for some $r > 0$, we have $Br(x_0) \subseteq \{x \in (0,1) : f(x) > a\}$, and the set is open.

   (c) Here we (must) assume $f : [0,1] \to \mathbb{R}$. For each $j = 0, 1, 2, \ldots, 2^n - 1$. Consider
   $$v_j = \inf_{x \in \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]} f(x).$$

We claim $v_j > -\infty$. Otherwise, there exists a sequence $\xi_j \in [0,1]$ with $f(\xi_j) \searrow -\infty$. By taking a subsequence, we may assume $\xi_j \to x_0 \in [0,1]$. By lower semicontinuity,

$$f(x_0) \leq \liminf_{x \to x_0} f(x) \leq \lim_{j \to \infty} f(\xi_j) = -\infty.$$

This contradicts the fact that f is real valued.

Observation: We have just shown that every lower semicontinuous function $f : [0,1] \to \mathbb{R}$ is bounded below. (See part (e).) Furthermore, replacing $-\infty$ with $m = \inf\{f(x) : x \in [0,1]\}$ in the reasoning above we find,

$$m \leq f(x_0) \leq \lim f(\xi_j) = m.$$

Thus, f attains its minimum value on $[0,1]$. (This is the solution to part (e).)

Returning to part (c), we know $v_j \in \mathbb{R}$, $j = 1, \ldots, 2^{n-1}$, and we may set

$$\bar{f}_n(x) = \begin{cases} v_1, & 0 \leq x < 1/2^n \\ v_j, & j/2^n < x < (j+1)/2^n, \quad j = 1, \ldots, 2^n - 2 \\ \min\{v_{j-1}, v_j\} & x = j/2^n, \qquad j = 1, \ldots, 2^n - 1 \\ v_{2^n - 1} & (2^n - 1)/2^n < x \leq 1. \end{cases}$$

This is clearly a sequence of lower semicontinuous step functions. Since

$$\begin{aligned} v_j &= \min\left\{ f(x) : x \in \left[\frac{2j}{2^{n+1}}, \frac{2j+1}{2^{n+1}}\right] \cup \left[\frac{2j+1}{2^{n+1}}, \frac{2j+2}{2^{n+1}}\right] \right\} \\ &\leq \min\left\{ f(x) : x \in \left[\frac{2j}{2^{n+1}}, \frac{2j+1}{2^{n+1}}\right] \right\}, \min\left\{ f(x) : x \in \left[\frac{2j+1}{2^{n+1}}, \frac{2j+2}{2^{n+1}}\right] \right\}, \end{aligned}$$

we easily see that $\bar{f}_1 \leq \bar{f}_2 \leq \bar{f}_3 \leq \cdots \leq f$. Furthermore, for any $\epsilon > 0$, there exists δ such that $|x - x_0| \leq \delta$, $x \in [0,1] \Rightarrow f(x) \geq f(x_0) - \epsilon$. Thus, for n large enough, $\bar{f}_n(x_0) \geq \min\{f(x) : |x - x_0| \leq \delta, x \in [0,1]\} \geq f(x_0) - \epsilon$. That is $\bar{f}_n(x_0) \to f(x_0)$.

Thus, we have shown that any lower semicontinuous $f : [0,1] \to \mathbb{R}$ is the limit of step functions as required.

On the other hand, if a sequence of lower semicontinuous functions $\bar{f}_j : [0,1] \to \mathbb{R}$ satisfy $\bar{f}_1 \leq \bar{f}_2 \leq \bar{f}_3 \leq \cdots \leq f : [0,1] \to \mathbb{R}$ with $\bar{f}_j \to f$ pointwise, then

$$\liminf_{x \to x_0} f(x) \geq \liminf_{x \to x_0} \bar{f}_j(x) \geq \bar{f}_j(x_0),$$

for every j. Thus, taking the limit as $j \to \infty$,

$$\liminf_{x \to x_0} f(x) \geq f(x_0),$$

i.e., f is lower semicontinuous. This also applies to one direction in part (d) since we didn't use the fact that the f_j were step functions.

(d) Let $\bar{f}_n$ be the step functions from part (c). Let f_n be linear on $\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]$ with $f_n(j/2^n) = \bar{f}_n(j/2^n)$, $j = 0, 1, \ldots, 2^n$:

$$f_n(x) = \bar{f}_n\left(\frac{j}{2^n}\right) + 2^n\left[\bar{f}_n\left(\frac{j+1}{2^n}\right) - \bar{f}_n\left(\frac{j}{2^n}\right)\right]\left(x - \frac{j}{2^n}\right) \quad \text{for } x \in \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right].$$

These functions are clearly continuous.
Also, $\bar{f}_{n+1}\left(\frac{2j+1}{2^{n+1}}\right) \geq f_n\left(\frac{2j+1}{2^{n+1}}\right)$. (The value $x = \frac{2j+1}{2^{n+1}}$ gives the new "middle" endpoints.) Since the values of $\bar{f}_{n+1}$ are also at least as great as f_n on the other endpoints, it follows that $f_1 \leq f_2 \leq f_3 \leq \cdots \leq f$. As before, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| \leq \delta$, $x \in [0, 1]$ implies $f(x) \geq f(x_0) - \epsilon$. Thus, for n large enough,

$$\begin{aligned} f_n(x_0) &\geq \min\left\{\bar{f}_n(x) : |x - x_0| \leq \delta, \quad x \in [0, 1]\right\} \\ &\geq \min\{f(x) : |x - x_0| \leq \delta, \quad x \in [0, 1]\} \\ &\geq f(x_0) - \epsilon. \end{aligned}$$

That is $f_n(x_0) \to f(x_0)$.
Note: A beautiful proof (without using step functions) was given by Sachin Jain:

$$f_n(x) = \inf\{f(t) + n|t - x| : t \in [0, 1]\}.$$

(I leave the details of checking that this works to you!)

(e) This is already done in part (c); let us simply note that the assumption the function is bounded is superfluous. This is not always the case if we assume $f : [0, 1] \to \mathbb{R} \cup \{\pm\infty\}$, where we would need to assume f is bounded below.

(f) Clearly, $g(x) \leq f(x)$, i.e., g is dominated by f.
We need to show

$$g(x) \leq \liminf_{\xi \to x} g(\xi) = \liminf_{\xi \to x}\left\{\sup_{\rho > 0} \inf_{|\eta - \xi| < \rho} f(\eta)\right\}.$$

Assume, by way of contradiction, that $\exists \epsilon > 0$ and $\exists x_j \to x$ with

$$\sup_{\rho > 0} \inf_{|\eta - \xi_j| < \rho} f(\eta) \leq g(x) - \epsilon.$$

Consequently, $\exists \rho_j \searrow 0$ and $\exists \eta_j$ such that $|\eta_j - \xi_j| < \rho_j$ with

$$f(\eta_j) \leq g(x) - \epsilon/2.$$

Since $|\eta_j - x| \leq |\eta_j - \xi_j| + |\xi_j - x|$, we see that $\eta_j \to x$. Hence,

$$g(x) \leq \liminf_{j \to \infty} f(\eta_j) \leq g(x) - \epsilon/2,$$

which is a contradiction.

Alternatively, one can argue directly that

$$\liminf_{\xi \to x} \left\{ \sup_{\rho > 0} \inf_{|\eta - \xi| < \rho} f(\eta) \right\} \geq \liminf_{\xi \to x} f(\xi) \geq g(x).$$

Either way, we see that g is lower semicontinuous.

Now, if $\tilde{f} \leq f$ is lower semicontinuous, then

$$\begin{aligned} \tilde{f}(x_0) &\leq \liminf_{x \to x_0} \tilde{f}(x) \\ &\leq \liminf_{x \to x_0} f(x) \\ &\leq \sup_{\epsilon > 0} \inf_{0 < |x - x_0| < \epsilon} f(x_0) \end{aligned}$$

We also know $\tilde{f}(x_0) \leq f(x_0)$. Therefore,

$$\tilde{f}(x_0) \leq \sup_{\epsilon > 0} \inf_{|x - x_0| < \epsilon} f(x_0) = g(x_0).$$

Notice that we have shown

$$\liminf_{x \to x_0} f(x) = \lim_{\epsilon \to 0} \inf_{0 < |x - x_0| < \epsilon} f(x) = \sup_{\epsilon} \inf_{0 < |x - x_0| < \epsilon} f(x).$$

3.

$$\begin{aligned} \{x : \{f_n(x)\} \text{ converges}\} &= \{x : \{f_n(x)\} \text{ is Cauchy}\} \\ &= \{x : \forall\ \epsilon > 0\ \exists\ N \text{ s.t. } n, m > N \Rightarrow |f_n(x) - f_m(x)| < \epsilon\} \\ &= \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m > N} \{x : |f_n(x) - f_m(x)| \leq 1/k\}. \end{aligned}$$

$\bigcap_{n,m>N} \{x : |f_n(x) - f_m(x)| \leq 1/k\}$ is closed.

5. (a) This is Theorem 2.10(a) in Folland. Thus, we obtain

$$\bar{f} = \sum_{j=1}^{k} a_j \chi_{E_j} \leq f.$$

(b) This is not possible. Take $\bar{f} = \chi_{[0,1] \setminus \mathbb{Q}}$. Then any nonnegative step function $\bar{g} \leq \bar{f}$ must be $\bar{g} \equiv 0$ (which differs from $\bar{f}$ on a set of full measure). What we can do, is assume the E_j are disjoint, the a_j are nonzero and distinct, and $E_0 = \{x : \bar{f}(x) = 0\}$. Then we can find (disjoint) closed sets $F_j \subseteq E_j$ s.t. $mF_j \geq mE_j - \epsilon/[2(k+1)]$ and open sets $U_j \supseteq E_j$ s.t. $mU_j \leq mE_j + \epsilon/[2(k+1)]$. With the closed sets, we note that for each $x \in (0,1) \setminus \cup F_j$, there is a unique largest interval $(a(x), b(x))$ with $x \in (a, b) \subseteq (0,1) \setminus \cup F_j$. Thus we may set

$$\tilde{f}_0(x) = \begin{cases} \bar{f}(x), & x \in \cup F_j \\ [\bar{f}(b) - \bar{f}(a)](x - a)/(b - a) + \bar{f}(a), & x \in (0,1) \setminus \cup F_j \end{cases}$$

to obtain a continuous function on $(0,1) \cup (\cup F_j)$ which satisfies $m\{x : \tilde{f}_0(x) \neq \bar{f}(x)\} \leq m([0,1] \backslash (\cup F_j)) = \sum (mE_j - mF_j) \leq \epsilon/2$.

Without loss of generality $U_i \cap F_j = \emptyset$ for $i \neq j$. (Replace U_i with $U_i \backslash \left(\bigcup_{j \neq i} F_j \right)$ if necessary.)

With the open sets, we can do the following. For each j, $U_j = \biguplus I_{j\ell}$ is a countable disjoint union of open intervals. (Let $\{q_\ell\} = U_j \cap \mathbb{Q}$; take I_{j1} to be the largest open interval in U_j which contains q_1. Obviously the endpoints of I_{j1} are not in U_j, or else we could make I_{j1} bigger. Thus, $U_j \backslash \bar{I}_{j1}$ is open. Let $j_2 = \min\{j : q_j \in U_j \backslash \bar{I}_{j1}\}$, and take I_{j2} to be the largest open interval in U_j which contains q_{j2}. Continue this process.) Since the closed sets $F_j \subset U_j$ are compact, we can take finitely many of the $I_{j\ell}$ such that first $\cup I_{j\ell} \supset Fj$ and $\sum mI_{j\ell} > mU_j - \epsilon/(2k)$.

Thus, we have finitely many intervals $\{I_{ij}\}$. Now let $I_1 = I_{11}$ and $I_2 = I_{12} \backslash \overline{I_{21}}$, $I_3 = I_{13} \backslash \left(\overline{I_1 \cup I_2} \right)$ etc. In this way we obtain finitely many disjoint intervals I_j and finitely many endpoints $\{x_\ell\}$ such that

$$\begin{aligned} I_1 \biguplus \cdots \biguplus I_{j_1} \bigcup \{x_\ell\} &\supseteq F_1 \\ I_{j_1+1} \biguplus \cdots \biguplus I_{j_2} \bigcup \{x_\ell\} &\supseteq F_2 \\ &\vdots \\ I_{j_{k-1}+1} \biguplus \cdots \biguplus I_{j_k} \bigcup \{x_\ell\} &\supseteq F_k. \end{aligned}$$

Thus we can form a step function

$$\bar{g} = \sum_{j=1}^{j_1} a_1 \chi_{I_j} + \sum_{j=j_1+1}^{j_2} a_2 \chi_{I_j} + \cdots + \sum_{j=j_{k-1}+1}^{j_k} a_k \chi_{I_k}$$

If $x \in \biguplus F_j \backslash \{x_\ell\}$, then $\bar{g}(x) = a_{j_0} = \bar{f}(x)$. Thus, $\{x : \bar{g}(x) \neq \bar{f}(x)\} \subseteq ([0,1] \backslash \biguplus F_j) \bigcup \{x_\ell\}$. Hence,

$$\begin{aligned} m\{x : \bar{g}(x) \neq \bar{f}(x)\} &\leq \sum_{j=1}^{k} m(E_j \backslash F_j) \\ &< \sum_{j=1}^{k} \frac{\epsilon}{2k} \\ &= \frac{\epsilon}{2}. \qquad \square \end{aligned}$$

(c) In this part, we can get $\tilde{f} \leq \bar{g}$, by setting

$$\tilde{f} = \sum \tilde{f}_j \chi_{I_j}$$

where

$$\tilde{f}_j = \begin{cases} m_j(x - \alpha_j), & \alpha_j \leq x \leq \alpha_j + \epsilon_j \\ m_j \epsilon_j, & \alpha_j + \epsilon \leq x \leq \beta_j - \epsilon \\ -m_j(x - \beta_j), & \beta_j - \epsilon \leq x \leq \beta_j, \end{cases}$$

$$
\begin{aligned}
I_j &= (\alpha_j, \beta_j), \\
\epsilon_j &= \min\left\{\frac{\epsilon}{4j_k}, \frac{\beta_j - \alpha_j}{2}\right\}, \qquad \text{and}
\end{aligned}
$$

$$
m_j = b_j/\epsilon_j = \begin{cases} a_1/\epsilon_1 & \text{for } 1 \le j \le j_1 \\ a_2/\epsilon_2 & \text{for } j_1 + 1 \le j \le j_2 \\ \vdots & \\ a_k/\epsilon_k & \text{for } j_{k-1} + 1 \le j \le j_k. \end{cases}
$$

Since the I_j are disjoint, $\tilde{f}|_{I_j} \le m_j \epsilon_j = b_j = \bar{g}|_{I_j}$ and $\tilde{f}|_{(\biguplus I_j)^c} \equiv 0 = \bar{g}|_{(\biguplus I_j)^c}$. Thus, $\tilde{f} \le \bar{g}$. Furthermore,

$$
\{x : \tilde{f}(x) \ne \bar{g}(x)\} \subseteq \left(\biguplus_{j=1}^{j_k} (\alpha_j, \alpha_j + \epsilon_j)\right) \bigcup \left(\biguplus_{j=1}^{j_k} (\beta_j - \epsilon_j, \beta_j)\right).
$$

So

$$
\begin{aligned}
m\{x : \tilde{f}(x) \ne \bar{g}(x)\} &\le \sum_{j=1}^{j_k} \epsilon_i + \sum_{j=1}^{i_k} \epsilon_j \\
&< 2 \sum_{j=1}^{j_k} \frac{\epsilon}{4j_k} \\
&= \frac{\epsilon}{2}.
\end{aligned}
$$

(d) (There was a typo in this problem. The f should be $\bar{f}$. See corrected exam. Actually, the problem could be modified in another way: Show that there is a nonnegative continuous function $\tilde{f}$ (not necessarily the one above) such that

$$
m\{x : \tilde{f}(x) \ne f(x)\} < \epsilon.
$$

This is true and is called Lusin's Theorem. But it is a bit harder; you can use Egeroff's Theorem.)

The intended problem is easy:

$$
\begin{aligned}
m\{x : \tilde{f}(x) \ne \bar{f}(x)\} &\le m\{x : \tilde{f}(x) \ne \bar{g}(x)\} + \{x : \bar{g}(x) \ne \bar{f}(x)\} \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}
$$

7. Follow the hint, let $\mathcal{C}_1$ be a maximal collection of disjoint balls in $\mathcal{C}$ with radii in $[R/2, R)$. That is

$$
\left.\begin{array}{l} \bar{B}_{\tilde{r}}(\tilde{x}) \in \mathcal{C} \backslash \mathcal{C}_1 \\ R/2 \le \tilde{r} < R \end{array}\right\} \Longrightarrow \bar{B}_{\tilde{r}}(\tilde{x}) \cap \bar{B}_r(x) \ne \emptyset \quad \text{for some } \bar{B}_r(x) \in \mathcal{C}_1.
$$

The collection $\mathcal{C}_1$ is clearly countable (since each ball contains a distinct point in $\mathbb{R}^n$ with rational coordinates), and $\bigcup_{\mathcal{C}_1} \bar{B}_{5r}(x) \supseteq \bigcup_{\tilde{\mathcal{C}}_1} \bar{B}_r(x)$ where $\tilde{\mathcal{C}}_1 = \{\bar{B}_{\tilde{r}}(\tilde{x}) \in$

$\mathcal{C} : R/2 \leq \tilde{r} < R\}$. To see the last assertion, assume $\bar{B}_{\tilde{r}}(\tilde{x}) \in \tilde{\mathcal{C}}_1$. Then $\exists\, x^* \in \bar{B}_{\tilde{r}}(\tilde{x}) \cap \bar{B}_r(x)$ for some $\bar{B}_r(x) \in \mathcal{C}_1$. Then for each $\xi \in \bar{B}_{\tilde{r}}(\tilde{x})$,

$$|\xi - x| \leq |\xi - \tilde{x}| + |\tilde{x} - x^*| + |x^* - x| \leq 2\tilde{r} + r \leq 2R + r \leq 5r.$$

Thus, $\bar{B}_{\tilde{r}}(\tilde{x}) \subseteq \bigcup_{\mathcal{C}_1} \bar{B}_{5r}(x)$. In fact, the same reasoning shows that *any* ball $\bar{B}_{\tilde{r}}(\tilde{x}) \in \mathcal{C}$ with $\tilde{r} \leq R$ which intersects $\bigcup_{\mathcal{C}_1} \bar{B}_r(x)$ is a subset of $\bigcup_{\mathcal{C}_1} \bar{B}_{5r}(x)$. Let $\mathcal{C}_2$ be a maximal collection of disjoint balls in $\{\bar{B}_r(x) \in \mathcal{C} : \bar{B}_r(x) \cap \bigcup_{\mathcal{C}_1} B_r(x) = \emptyset\}$ with radii in $[R/4, R/2)$. The reasoning above applies also in this case and shows that any ball in $\mathcal{C}$ with radius smaller than $R/2$ which intersects one of the balls in $\mathcal{C}_1 \cup \mathcal{C}_2$ is contained in the union of the expanded balls from $\mathcal{C}_1 \cup \mathcal{C}_2$ (with radius five times their usual radius). Furthermore, every ball in $\mathcal{C}$ with radius in $[R/4, R/2)$ must intersect one of the balls in $\mathcal{C}_1 \cup \mathcal{C}_2$. I trust you can see the rest of the argument from here.