Solutions and Notes for Exam 1

2. Probably the cleanest way to do this is via an alternative formulation of continuity:

f is continuous at x if for any $\epsilon > 0$, there exists $\delta > 0$ such that
$|\xi - x|, |\eta - x| < \delta \Rightarrow |f(\xi) - f(\eta)| < \epsilon$.

It is easy to see that this condition is equivalent to the usual one.

Note: The main advantage of this formulation is that it does not explicitly involve the value of $f(x)$. It is a curious fact that the analogous formulation for convergence of sequences which does not mention the limit (i.e., that involving the Cauchy condition) is standard in basic analysis, but one rarely finds the above formulation in elementary texts.

In any event, one sees that the continuity set is thus,

$$C = \bigcap_{n=1}^{\infty} \left\{ x : \exists\, \delta > 0 \text{ s.t. } |\xi - x|, |\eta - x| < \delta \Rightarrow |f(\xi) - f(\eta)| < \frac{1}{n} \right\}$$

and we are done if $A_n = \left\{ x : \exists\, \delta > 0 \text{ s.t. } |\xi - x|, |\eta - x| < \delta \Rightarrow |f(\xi) - f(\eta)| < \frac{1}{n} \right\}$ is open. In fact if $x \in A_n$, then $\exists\, \delta = \delta(x) > 0$ s.t. $|\xi - x|, |\eta - x| < \delta \Rightarrow |f(\xi) - f(\eta)| < 1/n$. Thus, $B_\delta(x) \subseteq A_n$, and A_n is open. (If $y \in B_\delta(x)$, then $\tilde{\delta} = \min\{x + \delta - y, y - x + \delta\} > 0$ and $|\xi - y| < \tilde{\delta} \Rightarrow y - \tilde{\delta} < \xi < y + \tilde{\delta} \Rightarrow x - \delta < \xi < x + \delta \Rightarrow |\xi - x| < \delta$. Similarly, $|\eta - y| < \tilde{\delta} \Rightarrow |\eta - x| < \delta$. So $|\xi - y|, |\eta - y| < \tilde{\delta} \Rightarrow |f(\xi) - f(\eta)| < 1/n$.)

Alternatively, one can follow Shuk-Yin Choi's use of the standard formulation to note that

$$C = \bigcup_{x \in C} \bigcap_{n=1}^{\infty} (x - \tilde{\delta}_{n,x}, x + \tilde{\delta}_{n,x}) \tag{1}$$

where $\tilde{\delta}_{n,x} = \min\{\delta_{n,x}, 1/n\}$ and $\delta_{n,x}$ is the standard δ for continuity ($|\xi - x| < \delta_{n,x} \Rightarrow |f(\xi) - f(x)| < 1/n$).

Note: One can replace $\tilde{\delta}_{n,x}$ by $\delta_{n,x}$ in (1), but then it requires proof that the resulting set is C. This way, we know $\tilde{\delta}_{n,x} \to 0$ (as $n \to \infty$), so the equality in (1) is obvious.

Then one makes the jump of faith and asserts

$$C = \bigcap_{n=1}^{\infty} \bigcup_{x \in C} (x - \tilde{\delta}_{n,x}, x + \tilde{\delta}_{n,x}) \tag{2}$$

which is obviously a G_δ. However, one must be careful as switching intersection/union order does not always give the same set. (Consider $\bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \left(\frac{1}{j} - \frac{1}{n}, \frac{1}{j} + \frac{1}{n} \right)$.) In any case, (2) turns out to be true — but it requires proof.

4. At least $A \subseteq \bigcup_{n=1}^{\infty}[-n, n]$, so m^* is well defined. (Just a technical point.)

$$0 \leq m^*\phi \leq \sum_{j=1}^{\infty} 0 = 0 \text{ since } \phi \subseteq [0, 0] \cup [0, 0] \cup \cdots.$$

(Or one could just say $m^*\phi = 0$ is "obvious.") We need monotonicity and subadditivity.

Monotonicity: If $A \subseteq B$ and $B \subseteq \bigcup[c_j, b_j]$, then $A \subseteq \bigcup[a_j, b_j]$. Therefore, $m^*A \leq m^*B$.

Subadditivity: If $m^*A_j = \infty$ for any j, then there is nothing to prove. Otherwise, for any $\epsilon > 0$, $\exists [a_j^k, b_j^k]$ s.t. $A_j \subseteq \bigcup_k[a_j^k, b_j^k]$ and $m^*A_j \geq \sum_k(b_j^k - a_j^k) - \frac{\epsilon}{2^j}$. (In fact, this is true even if $m^*A_j = \infty$.) In any case, $\bigcup_j A_j \subseteq \bigcup_{j,k}[a_j^k, b_j^k]$ and so

$$m^*\left(\bigcup_j A_j\right) \leq \sum_{j,k}(b_j^k - a_j^k) \leq \sum_j m^*A_j + \epsilon.$$

Since ϵ is arbitrary, we are done. $\quad\square$

If $m^*Z = 0$, then $m^*(A \cup Z) \leq m^*A + m^*Z = m^*A$ by subadditivity. The reverse inequality is true by monotonicity: $m^*A \leq m^*(A \cup Z)$ since $A \subseteq A \cup Z$. $\quad\square$

6. Assume by way of contradiction (BWOC) that

$$\mu\{x : f(x) > 0\} > 0. \tag{3}$$

Let $A_j = \{x : f(x) > 1/j\}$ for $j = 1, 2, 3 \dots$. Then, since $\{x : f(x) > 0\} = \bigcup_{j=1} A_j$ and $A_1 \subseteq A_2 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, we have

$$0 < \mu\{x : f(x) > 0\} = \lim_{j \to \infty} \mu A_j$$

by Theorem 1.8c. Therefore, $\mu A_j > 0$ for some j. Thus, $f > \frac{1}{j}\chi_{A_j}$, and by the definition of $\int f$,

$$\int f \geq \frac{1}{j}\mu A_j > 0.$$

This contradicts the fact that $\int f = 0$ and, hence, our assumption (3).