Name: ______________________

Math 6327 Real Analysis — Exam 1(Takehome)

1. A function $f : (0,1) \to \mathbb{R}$ is called *lower semicontinuous at* $x_0$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x).$$

**Correction:** We should say that a function $f : A \to \mathbb{R}$ is called *lower semicontinuous at* $x_0$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x)$$

where $A$ is any subest of $\mathbb{R}$ and

$$\liminf_{x \to x_0} f(x) := \lim_{\delta \to 0} \inf_{0<|x-x_0|<\delta} f(x).$$

Of course, in the limit, we only consider points $x \in A$. It can be an exercise to show this is the same as

$$\sup_{\delta>0} \inf_{0<|x-x_0|<\delta} f(x).$$

The function is *lower semicontinuous* if it is lower semicontinuous at ever point. Similarly, $f$ is *upper semicontinuous* if

$$f(x_0) \geq \limsup_{x \to x_0} f(x).$$

**Correction:** For parts (a-b) below, we assume $A = (0,1)$. For parts (c-f) we assume $A = [0,1]$.

(a) Show that $f$ is continuous at a point if and only if it is both upper and lower semicontinuous at the point.

(b) Show that every lower semicontinuous function is Borel measurable.

(c) Show that a function is lower semicontinuous if and only if there is a sequence of lower semicontinuous *step functions* $\bar{f}_j$ with $\bar{f}_1 \leq \bar{f}_2 \leq \bar{f}_3 \leq \cdots$ and

$$\lim_{j \to \infty} \bar{f}_j = f.$$

(A *step function* is one for which there are finitely many points $0 = x_0 < x_2 < x_3 < \cdots < x_k = 1$ such that $\bar{f}$ is constant on each open interval $(x_j, x_{j+1})$.)

(d) Show that a function $f$ is lower semicontinuous if and only if there is an increasing sequence of continuous functions which converge to $f$.

(e) Show that a lower semicontinuous function which is bounded from below attains its minimum value on $[0,1]$.

(f) Let $f$ be a bounded function on $[0,1]$ and set

$$g(x) = \sup_{r>0} \inf_{|\xi - x|<r} f(\xi).$$

Show that $g$ is lower semicontinuous and that $g$ is the largest lower semicontinuous function dominated by $f$, i.e., if $\tilde{f} \le f$ is lower semicontinuous, then $\tilde{f} \le g$.

2. If $f : \mathbb{R} \to \mathbb{R}$, then the set of points of continuity of $f$ is a $G_\delta$.

3. If $f_1, f_2, f_3, \ldots$ is any sequence of continuous functions defined on $\mathbb{R}$, then the set of points at which the sequence converges is an $F_{\sigma\delta}$.

4. For $A \subset \mathbb{R}$ define

$$m^* A = \inf_{A \subset \cup [a_j, b_j]} \sum (b_j - a_j)$$

where the infemum is taken over countable unions of closed intervals containing $A$.

Show that $m^*$ is an outer measure, and that

$$m^* Z = 0 \qquad \Longrightarrow \qquad m^*(A \cup Z) = m^* A$$

for any set $A \subset \mathbb{R}$.

5. Let $f$ be a nonnegative Lebesgue measurable function on $[0,1]$ which is bounded. Let $\epsilon > 0$.

(a) There is a nonnegative simple function $\bar{f} \le f$ such that

$$|f(x) - \bar{f}(x)| < \epsilon$$

for all $x$.

(b) There is a nonnegative step function $\bar{g} \le \bar{f}$ such that

$$m\{x : \bar{g}(x) \ne \bar{f}(x)\} < \epsilon/2.$$

Hint: Use the fact that $m$ is Borel regular.

**Correction:** Part (b) is impossible for $f = \bar{f} = \chi_{[0,1]\setminus\mathbb{Q}}$. Forget about $\bar{g} \le \bar{f}$.

(c) There is a nonnegative continuous function $\tilde{f} \le \bar{g}$ such that

$$m\{x : \tilde{f}(x) \ne \bar{g}(x)\} < \epsilon/2.$$

(d) $m\{x : \tilde{f}(x) \ne f(x)\} < \epsilon$.

**Correction:** Part (d) should read "$m\{x : \tilde{f}(x) \ne \bar{f}(x)\} < \epsilon$." Actually, it's true that you *can* find a continuous function which agrees with $f$ except on a set of arbitrarily small measure. That's called Lusin's Theorem, but it's a bit harder than what I had in mind here. A good idea for the final.

6. Let $f$ be a nonnegative measureable function on $\mathbb{R}$. Show that

$$\int f = 0 \qquad \Longrightarrow \qquad f = 0 \text{ a.e.}$$

7. Let $\bar{B}_r(x_0) = \{x : |x - x_0| \leq r\}$ denote the closed ball of radius $r$ in $\mathbb{R}^n$ centered at $x_0$. Let $\mathcal{C}$ be a collection of such closed balls with bounded radii, i.e., there is some $R > 0$ such that $r < R$ for every $\bar{B}_r(x) \in \mathcal{C}$.

   Show that there is a countable disjoint subcollection of the balls $\{\bar{B}_{r_j}(x_j)\}$ in $\mathcal{C}$ such that

   $$\bigcup_{\mathcal{C}} \bar{B}_r(x) \subset \bigcup_j \bar{B}_{5r_j}(x_j).$$

   Hint: Consider the biggest radii $R/2 \leq r < R$ first, and take a maximal collection of disjoint balls. Then consder the "next smaller radii" $R/4 \leq r < R/2$ etc..