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Math 6327 Real Analysis

Exam 1(Takehome)

1. A function $f:(0,1) \to \mathbb{R}$ is called *lower semicontinuous at* x_0 if

$$f(x_0) \le \liminf_{x \to x_0} f(x).$$

The function is *lower semicontinuous* if it is lower semicontinuous at ever point. Similarly, f is *upper semicontinuous* if

$$f(x_0) \ge \limsup_{x \to x_0} f(x).$$

- (a) Show that f is continuous at a point if and only if it is both upper and lower semicontinuous at the point.
- (b) Show that every lower semicontinuous function is Borel measurable.
- (c) Show that a function is lower semicontinuous if and only if there is a sequence of lower semicontinuous step functions \bar{f}_j with $\bar{f}_1 \leq \bar{f}_2 \leq \bar{f}_3 \leq \cdots$ and

$$\lim_{j \to \infty} \bar{f}_j = f.$$

(A step function is one for which there are finitely many points $0 = x_0 < x_2 < x_3 < \cdots < x_k = 1$ such that \overline{f} is constant on each open interval (x_j, x_{j+1}) .)

- (d) Show that a function f is lower semicontinuous if and only if there is an increasing sequence of continuous functions which converge to f.
- (e) Show that a lower semicontinuous function which is bounded from below attains its minimum value on [0, 1].
- (f) Let f be a bounded function on [0, 1] and set

$$g(x) = \sup_{r>0} \inf_{|\xi-x| < r} f(\xi).$$

Show that g is lower semicontinuous and that g is the largest lower semicontinuous function dominated by f, i.e., if $\tilde{f} \leq f$ is lower semicontinuous, then $\tilde{f} \leq g$.

- 2. If $f : \mathbb{R} \to \mathbb{R}$, then the set of points of continuity of f is a G_{δ} .
- 3. If f_1, f_2, f_3, \ldots is any sequence of continuous functions defined on \mathbb{R} , then the set of points at which the sequence converges is an $F_{\sigma\delta}$.
- 4. For $A \subset \mathbb{R}$ define

$$m^*A = \inf_{A \subset \cup [a_j, b_j]} \sum (b_j - a_j)$$

where the infemum is taken over countable unions of closed intervals containing A. Show that m^* is an outer measure, and that

$$m^*Z = 0 \implies m^*(A \cup Z) = m^*A$$

for any set $A \subset \mathbb{R}$.

- 5. Let f be a nonnegative Lebesgue measurable function on [0, 1] which is bounded. Let $\epsilon > 0$.
 - (a) There is a nonnegative simple function $\bar{f} \leq f$ such that

$$|f(x) - \bar{f}(x)| < \epsilon$$

for all x.

(b) There is a nonnegative step function $\bar{g} \leq \bar{f}$ such that

$$m\{x: \bar{g}(x) \neq \bar{f}(x)\} < \epsilon/2.$$

Hint: Use the fact that m is Borel regular.

(c) There is a nonnegative continuous function $\tilde{f} \leq \bar{g}$ such that

$$m\{x: \hat{f}(x) \neq \bar{g}(x)\} < \epsilon/2.$$

(d) $m\{x: \tilde{f}(x) \neq f(x)\} < \epsilon.$

6. Let f be a nonnegative measureable function on \mathbb{R} . Show that

$$\int f = 0 \qquad \Longrightarrow \qquad f = 0 \text{ a.e}$$

7. Let $\bar{B}_r(x_0) = \{x : |x - x_0| \le r\}$ denote the closed ball of radius r in \mathbb{R}^n centered at x_0 . Let \mathcal{C} be a collection of such closed balls with bounded radii, i.e., there is some R > 0 such that r < R for every $\bar{B}_r(x) \in \mathcal{C}$.

Show that there is a countable disjoint subcollection of the balls $\{\bar{B}_{r_j}(x_j)\}$ in \mathcal{C} such that

$$\bigcup_{\mathcal{C}} \bar{B}_r(x) \subset \bigcup_j \bar{B}_{5r_j}(x_j).$$

Hint: Consider the biggest radii $R/2 \le r < R$ first, and take a maximal collection of disjoint balls. Then consder the "next smaller radii" $R/4 \le r < R/2$ etc..